

Approximate Correctors and Convergence Rates in Almost-Periodic Homogenization

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Abstract

We carry out a comprehensive study of quantitative homogenization of second-order elliptic systems with bounded measurable coefficients that are almost-periodic in the sense of H. Weyl. We obtain uniform local L^2 estimates for the approximate correctors in terms of a function that quantifies the almost-periodicity of the coefficient matrix. We give a condition that implies the existence of (true) correctors. These estimates as well as similar estimates for the dual approximate correctors yield optimal or near optimal convergence rates in H^1 and L^2 . The L^2 -based Hölder and Lipschitz estimates at large scale are also established.

Keywords. Homogenization; Almost Periodic; Approximate Correctors; Convergence Rates.

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1 Introduction

In this paper we shall be interested in quantitative homogenization of a family of second-order elliptic operators with rapidly oscillating almost-periodic coefficients,

$$\mathcal{L}_\varepsilon = -\operatorname{div}(A(x/\varepsilon)\nabla) = -\frac{\partial}{\partial x_i} \left\{ a_{ij}^{\alpha\beta} \left(\frac{x}{\varepsilon} \right) \frac{\partial}{\partial x_j} \right\}, \quad \varepsilon > 0 \quad (1.1)$$

(the summation convention is used throughout). We assume that the coefficient matrix $A(y) = (a_{ij}^{\alpha\beta}(y))$ with $1 \leq i, j \leq d$ and $1 \leq \alpha, \beta \leq m$ is real, bounded measurable, and satisfies the ellipticity condition,

$$\mu |\xi|^2 \leq a_{ij}^{\alpha\beta}(y) \xi_i^\alpha \xi_j^\beta \leq \mu^{-1} |\xi|^2 \quad \text{for a.e. } y \in \mathbb{R}^d \text{ and } \xi = (\xi_i^\alpha) \in \mathbb{R}^{m \times d}, \quad (1.2)$$

where $\mu > 0$. We further assume that $A(y)$ is almost-periodic (a.p.) in sense of H. Weyl, which we denote by $A \in APW^2(\mathbb{R}^d)$. This means that each entry of A may be approximated by a sequence of real trigonometric polynomials with respect to the semi-norm,

$$\|F\|_{W^2} := \limsup_{R \rightarrow \infty} \sup_{x \in \mathbb{R}^d} \left(\int_{B(x,R)} |F|^2 \right)^{1/2}. \quad (1.3)$$

The qualitative homogenization theory for elliptic equations and systems with a.p. coefficients has been known since late 1970's [13, 15]. Let $u_\varepsilon \in H^1(\Omega; \mathbb{R}^m)$ be the weak solution to the Dirichlet problem,

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = f \quad \text{on } \partial\Omega, \quad (1.4)$$

where $F \in H^{-1}(\Omega; \mathbb{R}^m)$, $f \in H^{1/2}(\partial\Omega; \mathbb{R}^m)$, and Ω is a bounded Lipschitz domain in \mathbb{R}^d . Suppose that $A(y)$ satisfies the ellipticity condition (1.2) and is a.p. in the sense of Besicovich (a larger class than $APW^2(\mathbb{R}^d)$). Then u_ε converges weakly in $H^1(\Omega; \mathbb{R}^m)$ and thus strongly in $L^2(\Omega; \mathbb{R}^m)$ to a function $u_0 \in H^1(\Omega; \mathbb{R}^m)$. Moreover, u_0 is the weak solution to the (homogenized) Dirichlet problem,

$$\mathcal{L}_0(u_0) = F \quad \text{in } \Omega \quad \text{and} \quad u_0 = f \quad \text{on } \partial\Omega, \quad (1.5)$$

where \mathcal{L}_0 is a second-order elliptic operator with constant coefficients that depend only on A . Our primary interest in this paper is in the convergence rates for $\|u_\varepsilon - u_0\|_{L^2(\Omega)}$.

In the case that A is uniformly a.p. (almost-periodic in the sense of H. Bohr), the problem of convergence rates and uniform Hölder estimates for the Dirichlet problem (1.4) were studied recently by the first author in [17] (also see earlier work [13, 9, 7] as well as [11, 14, 8] for homogenization of nonlinear differential equations in the a.p. setting). The results in [17] were subsequently used by S.N. Armstrong and the first author in [3] to establish the uniform Lipschitz estimates, up to the boundary, for solutions of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ with either Dirichlet or Neumann conditions. In particular, it follows from [3] that the so-called approximate correctors χ_T , defined in (1.6) below, satisfy the uniform Lipschitz estimate $\|\nabla \chi_T\|_\infty \leq C$, if A is Hölder continuous and satisfies an almost-periodicity condition. Under some additional assumptions, the uniform boundedness of χ_T , $\|\chi_T\|_\infty \leq C$ and the existence of (true) correctors were obtained recently in [1].

In this paper we carry out a comprehensive study of quantitative homogenization of second-order elliptic systems with coefficients in $APW^2(\mathbb{R}^d)$. Our results improve and

extend those in [17, 1] to a much broader class of a.p. functions, which allows bounded measurable coefficients (for comparison, uniformly a.p. functions are uniformly continuous in \mathbb{R}^d). Notice that the semi-norm $\|F\|_{W^2}$ in (1.3) is translation and dilation invariant. As such the class of coefficients $A \in APW^2(\mathbb{R}^d)$ seems to be a natural choice for studying quantitative properties in a.p. homogenization without smoothness assumptions.

As in the case of uniformly a.p. (or random) coefficients, to obtain the convergence rates, the key step is to establish estimates for the approximate correctors χ_T , defined by the elliptic system

$$-\operatorname{div}(A\nabla\chi_T) + T^{-2}\chi_T = \operatorname{div}(A\nabla P), \quad (1.6)$$

where $T \geq 1$ and P is an affine function. To quantify the almost-periodicity of the coefficient matrix A , we introduce a function $\rho_k(L, R)$, defined by (2.18) in Section 2. It is known that a bounded function A is a.p. in the sense of H. Weyl if and only if $\rho_1(L, R) \rightarrow 0$ as $L, R \rightarrow \infty$ (see Section 2). We remark that the function ρ_1 , which only involves the first-order difference, was used in [17]. Our definition of the higher-order version $\rho_k(L, R)$, as well as one of main steps in the proof of Theorems 1.1 and 1.2, is inspired by [1], where a similar function was used to give a sufficient condition for the existence of (true) correctors.

The following is one of main results of the paper.

Theorem 1.1. *Suppose that $A \in APW^2(\mathbb{R}^d)$ and satisfies the ellipticity condition (1.2). Fix $k \geq 1$ and $\sigma \in (0, 1)$. Then there exists a constant $c > 0$, depending only on d and k , such that for any $T \geq 2$,*

$$\|\nabla\chi_T\|_{S_1^2} \leq C_\sigma T^\sigma, \quad (1.7)$$

and

$$\|\chi_T\|_{S_1^2} \leq C_\sigma \int_1^T \inf_{1 \leq L \leq t} \left\{ \rho_k(L, t) + \exp\left(-\frac{ct^2}{L^2}\right) \right\} \left(\frac{T}{t}\right)^\sigma dt, \quad (1.8)$$

where C_σ depends only on σ , k and A .

In the theorem above we have used the notation

$$\|F\|_{S_R^p} := \sup_{x \in \mathbb{R}^d} \left(\int_{B(x, R)} |F|^p \right)^{1/p} \quad (1.9)$$

for $1 \leq p < \infty$ and $0 < R < \infty$. Since $\rho_k(L, R) \rightarrow 0$ as $L, R \rightarrow \infty$, it follows from (1.8) that $T^{-1}\|\chi_T\|_{S_1^2} \rightarrow 0$, as $T \rightarrow \infty$. In particular, if there exist some $k \geq 1$ and $\alpha \in (0, 1]$ such that

$$\rho_k(L, L) \leq CL^{-\alpha} \quad \text{for any } L \geq 1, \quad (1.10)$$

then $\|\chi_T\|_{S_1^2} \leq C_\beta T^\beta$ for any $\beta > 1 - \alpha$. Our next two theorems provide sufficient conditions for the existence of true correctors in $APW^2(\mathbb{R}^d)$ and for the boundedness of $\|\nabla\chi_T\|_{S_1^2}$, respectively.

Theorem 1.2. *Suppose A satisfies the same conditions as in Theorem 1.1. Also assume that there exist some $k \geq 1$ and $\alpha > 1$ such that (1.10) holds. Then $\|\chi_T\|_{S_1^2} \leq C$. Moreover, for each affine function P , the system for the (true) corrector*

$$-\operatorname{div}(A\nabla\chi) = \operatorname{div}(A\nabla P) \quad \text{in } \mathbb{R}^d$$

has a weak solution χ such that $\chi, \nabla\chi \in APW^2(\mathbb{R}^d)$.

Theorem 1.3. *Suppose A satisfies the same conditions as in Theorem 1.1. Also assume that there exist some $k \geq 1$ and $\alpha > 3$ such that*

$$\rho_k(L, L) \leq C \{\log(L)\}^{-\alpha} \quad \text{for any } L \geq 2. \quad (1.11)$$

Then for any $T \geq 1$,

$$\|\nabla \chi_T\|_{S_1^2} \leq C, \quad (1.12)$$

where C is independent of T .

Using the estimates in Theorems 1.1 and 1.2 for the approximate correctors χ_T as well as similar estimates for the dual approximate correctors, we are able to establish a convergence rate in $L^2(\Omega; \mathbb{R}^m)$ under the condition that $u_0 \in H^2(\Omega; \mathbb{R}^m)$. In the following theorem, the function ψ , defined in (2.4), is the limit of $\nabla \chi_T$ in $B^2(\mathbb{R}^d)$, while ψ^* and χ_T^* are the corresponding functions for the adjoint operator $\mathcal{L}_\varepsilon^*$. Also, we use $\Theta_{k,\sigma}(T)$ to denote the integral in the r.h.s of (1.8).

Theorem 1.4. *Suppose A satisfies the same conditions as in Theorem 1.1. Let Ω be a bounded $C^{1,1}$ domain in \mathbb{R}^d . Let u_ε, u_0 be weak solutions of (1.4) and (1.5), respectively, in Ω . Assume further that $u_0 \in H^2(\Omega; \mathbb{R}^m)$. Then, for any $0 < \varepsilon < 1$,*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C_\sigma \left\{ \|\nabla \chi_T - \psi\|_{B^2} + \|\nabla \chi_T^* - \psi^*\|_{B^2} + T^{-1} \Theta_{k,\sigma}(T) \right\} \|u_0\|_{H^2(\Omega)}, \quad (1.13)$$

where $T = \varepsilon^{-1}$. The constant C_σ depends only on σ, k, A and Ω . Furthermore, if (1.10) holds for some $\alpha > 1$ and $k \geq 1$, then

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C \varepsilon \|u_0\|_{H^2(\Omega)}. \quad (1.14)$$

We now describe the outline of this paper and some of key ideas used in the proof of Theorems 1.1-1.4. In Section 2 we provide a brief review of the qualitative homogenization theory of second-order elliptic systems with coefficients that are a.p. in the sense of Besicovitch. In Sections 3 and 4 we introduce the approximate correctors χ_T and establish some preliminary estimates for χ_T . Using Tartar's method of test functions as well as the estimates of χ_T obtained in Section 4, we prove a compactness theorem in Section 5 on a sequence of elliptic operators $\mathcal{L}_{\varepsilon_\ell}^\ell + \lambda_\ell = -\operatorname{div}(A^\ell(x/\varepsilon_\ell)\nabla) + \lambda_\ell$, where each $A^\ell(y)$ is obtained from $A(y)$ through a translation. With this compactness theorem at our disposal, an L^2 -based Hölder estimates at large scale for solutions of $\mathcal{L}_\varepsilon(u_\varepsilon) + \lambda u_\varepsilon = F + \operatorname{div}(f)$ are obtained in Section 6. This is done by using a compactness argument, introduced to the study of homogenization problems by Avellaneda and Lin [5]. As a corollary of the Hölder estimates at large scale, we obtain the estimate (1.7) as well as a Liouville property for solutions of $\mathcal{L}_1(u) = 0$ in \mathbb{R}^d .

In Section 7 we establish some general estimates for functions g in $APW^2(\mathbb{R}^d)$. These estimates, which formalize and extend a quantitative ergodic argument in [1], allow us to control the norm $\|g\|_{S_1^2}$ by $\|\nabla g\|_{S_t^2}$ for $t \geq 1$ and the higher-order differences of ∇g (see Theorem 7.3). The estimate (1.8) in Theorem 1.1 as well as Theorem 1.2 is proved in Section 8 by combining estimates in Section 7 and the large-scale Hölder estimates in Section 6. In particular, the existence of correctors in $APW^2(\mathbb{R}^d)$ under the condition (1.10) for some $\alpha > 1$ is obtained by showing that

$$\|\chi_T - \chi_{2T}\|_{S_1^2} \leq C T^{-\beta},$$

for all $T \geq 1$ and some $\beta > 0$. In Section 9 we introduce the dual approximate correctors ϕ_T , defined by the elliptic system

$$-\Delta \phi_T + T^{-2} \phi_T = b_T - \langle b_T \rangle, \quad (1.15)$$

where $b_T = A + A \nabla \chi_T - \langle A \rangle$. Estimates for ϕ_T and their derivatives are obtained by using a line of argument similar to that for χ_T . In particular, we show that

$$T^{-1} \|\phi_T\|_{S_1^2} + \|\nabla \phi_T\|_{S_1^2} \leq C_\sigma \Theta_{k,\sigma}(T), \quad (1.16)$$

for any $\sigma \in (0, 1)$ and $T \geq 2$. Using the estimates for χ_T and ϕ_T , we give the proof of Theorem 1.4 in Section 10. To do this we adapt a line of argument for establishing sharp L^2 convergence rates in periodic homogenization in [16, 18], which was motivated by the approach used in [19]. The idea is to first establish the following error estimate in H^1 ,

$$\begin{aligned} & \|u_\varepsilon - u_0 - \varepsilon \chi_T(x/\varepsilon) K_{\varepsilon,\delta}(\nabla u_0)\|_{H^1(\Omega)} \\ & \leq C \left\{ \|\nabla \chi_T - \psi\|_{B^2} + T^{-1} \Theta_{k,\sigma}(T) \right\}^{1/2} \|u_0\|_{H^2(\Omega)}, \end{aligned} \quad (1.17)$$

where $T = \varepsilon^{-1}$ and $K_{\varepsilon,\delta}$ is a smoothing operator defined by $K_{\varepsilon,\delta}(f) = \xi_\varepsilon * (\eta_\delta f)$. The desired estimate for $\|u_\varepsilon - u_0\|_{L^2(\Omega)}$ follows from (1.17) by a duality argument. In Section 10 we formalize this approach in the a.p. setting so that further improvement on the estimates of approximate and dual approximate correctors automatically leads to improvement on the rate of convergence in L^2 .

Finally, Theorem 1.3 is proved in Section 11. To do this, we first establish an L^2 -based large-scale Lipschitz estimate for $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ under the condition (1.11) for some $k \geq 1$ and $\alpha > 3$. The proof, which uses the convergence rates in Theorem 1.4, is based on an approach developed in [4] and further improved in [3, 2, 16]. Estimate (1.12) follows readily from the L^2 -based Lipschitz estimate.

Throughout this paper we will use $\int_E f = \frac{1}{|E|} \int_E f$ to denote the L^1 average of a function f over a set E , and C to denote constants that depend at most on A , Ω and other relevant parameters, but never on ε or T .

2 Almost-periodic homogenization

In this section we give a brief review of the qualitative homogenization theory for elliptic systems with a.p. coefficients. A detailed presentation may be found in [12].

Let $\text{Trig}(\mathbb{R}^d)$ denote the set of real trigonometric polynomials in \mathbb{R}^d . A function f in $L_{\text{loc}}^2(\mathbb{R}^d)$ is said to belong to $B^2(\mathbb{R}^d)$ if f is the limit of a sequence of functions in $\text{Trig}(\mathbb{R}^d)$ with respect to the semi-norm

$$\|f\|_{B^2} := \limsup_{R \rightarrow \infty} \left(\int_{B(0,R)} |f|^2 \right)^{1/2}. \quad (2.1)$$

Functions in $B^2(\mathbb{R}^d)$ are said to be a.p. in the sense of Besicovitch. It is not hard to see that if $g \in L^\infty(\mathbb{R}^d) \cap B^2(\mathbb{R}^d)$ and $f \in B^2(\mathbb{R}^d)$, then $fg \in B^2(\mathbb{R}^d)$.

Let $f \in L_{\text{loc}}^1(\mathbb{R}^d)$. A number $\langle f \rangle$ is called the mean value of f if

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^d} f(x/\varepsilon) \varphi(x) dx = \langle f \rangle \int_{\mathbb{R}^d} \varphi \quad (2.2)$$

for any $\varphi \in C_0^\infty(\mathbb{R}^d)$. It is known that if $f, g \in B^2(\mathbb{R}^d)$, then fg has a mean value. Under the equivalent relation that $f \sim g$ if $\|f - g\|_{B^2} = 0$, the set $B^2(\mathbb{R}^d)$ becomes a Hilbert space with the inner product defined by $(f, g) = \langle fg \rangle$. Furthermore, if $B^2(\mathbb{R}^d; \mathbb{R}^k) = B^2(\mathbb{R}^d) \times \cdots \times B^2(\mathbb{R}^d)$, then the following Weyl's orthogonal decomposition

$$B^2(\mathbb{R}^d; \mathbb{R}^{m \times d}) = V_{\text{pot}}^2 \oplus V_{\text{sol}}^2 \oplus \mathbb{R}^{m \times d} \quad (2.3)$$

holds, where V_{pot}^2 (resp., V_{sol}^2) denotes the closure of potential (resp., solenoidal) trigonometric polynomials with mean value zero in $B^2(\mathbb{R}^d; \mathbb{R}^{m \times d})$.

Suppose that $A = (a_{ij}^{\alpha\beta})$ satisfies the ellipticity condition (1.2) and $A \in B^2(\mathbb{R}^d)$, i.e., each entry $a_{ij}^{\alpha\beta} \in B^2(\mathbb{R}^d)$. For each $1 \leq j \leq d$ and $1 \leq \beta \leq m$, let $\psi_j^\beta = (\psi_{ij}^{\alpha\beta})$ be the unique function in V_{pot}^2 such that

$$(a_{ik}^{\alpha\gamma} \psi_{kj}^{\gamma\beta}, \phi_i^\alpha) = -(a_{ij}^{\alpha\beta}, \phi_i^\alpha) \quad \text{for any } \phi = (\phi_i^\alpha) \in V_{\text{pot}}^2. \quad (2.4)$$

Let $\widehat{A} = (\widehat{a}_{ij}^{\alpha\beta})$ be the homogenized matrix of A , where

$$\widehat{a}_{ij}^{\alpha\beta} = \langle a_{ij}^{\alpha\beta} \rangle + \langle a_{ik}^{\alpha\gamma} \psi_{kj}^{\gamma\beta} \rangle. \quad (2.5)$$

Using

$$\widehat{a}_{ij}^{\alpha\beta} = (a_{lk}^{t\gamma} (\psi_{kj}^{\gamma\beta} + \delta_{kj} \delta^{\gamma\beta}), \psi_{li}^{t\alpha} + \delta_{li} \delta^{t\alpha}), \quad (2.6)$$

where we have used δ_{ij} and $\delta^{\alpha\beta}$ for Kronecker's delta, it can be proved that

$$\mu |\xi|^2 \leq \widehat{a}_{ij}^{\alpha\beta} \xi_i^\alpha \xi_j^\beta \leq \mu_1 |\xi|^2 \quad (2.7)$$

for any $\xi = (\xi_i^\alpha) \in \mathbb{R}^{m \times d}$, where μ_1 depends only on d, m and μ . Moreover, $\widehat{A}^* = (\widehat{A})^*$, where A^* denotes the adjoint of A .

The next theorem, whose proof may be found in [12], shows that the homogenized operator for \mathcal{L}_ε is given by $\mathcal{L}_0 = -\text{div}(\widehat{A}\nabla)$.

Theorem 2.1. *Suppose that A is real, bounded measurable, and satisfies (1.2). Also assume that $A \in B^2(\mathbb{R}^d)$. Let Ω be a bounded Lipschitz domain in \mathbb{R}^d and $F \in H^{-1}(\Omega; \mathbb{R}^m)$. Let $u_{\varepsilon_\ell} \in H^1(\Omega; \mathbb{R}^m)$ be a weak solution of $\mathcal{L}_{\varepsilon_\ell}(u_{\varepsilon_\ell}) = F$ in Ω , where $\varepsilon_\ell \rightarrow 0$. Suppose that $u_{\varepsilon_\ell} \rightharpoonup u_0$ weakly in $H^1(\Omega; \mathbb{R}^m)$. Then $A(x/\varepsilon_\ell)\nabla u_{\varepsilon_\ell} \rightharpoonup \widehat{A}\nabla u_0$ weakly in $L^2(\Omega; \mathbb{R}^{m \times d})$. Consequently, if $f \in H^{1/2}(\partial\Omega; \mathbb{R}^m)$ and u_ε is the weak solution to the Dirichlet problem:*

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = f \quad \text{on } \partial\Omega, \quad (2.8)$$

then, as $\varepsilon \rightarrow 0$, $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1(\Omega; \mathbb{R}^m)$, where u_0 is the weak solution to

$$\mathcal{L}_0(u_0) = F \quad \text{in } \Omega \quad \text{and} \quad u_0 = f \quad \text{on } \partial\Omega. \quad (2.9)$$

Let $P_j^\beta(x) = x_j e^\beta$, where $e^\beta = (0, \dots, 1, \dots, 0)$ with 1 in the β^{th} position. In the periodic case the homogenized coefficients $\widehat{a}_{ij}^{\alpha\beta}$ are obtained by solving the elliptic system

$$-\text{div}(A(x)\nabla u) = \text{div}(A(x)\nabla P_j^\beta), \quad (2.10)$$

with periodic boundary conditions in a periodic cell Y and then extending the solutions to \mathbb{R}^d by periodicity. Let $\chi_j^\beta = (\chi_j^{1\beta}, \dots, \chi_j^{m\beta})$ denote the periodic solution of (2.10) in \mathbb{R}^d such that $\int_Y \chi_j^\beta = 0$. Such solutions are called the correctors for \mathcal{L}_ε . Then

$$\widehat{a}_{ij}^{\alpha\beta} = \oint_Y \left\{ a_{ij}^{\alpha\beta} + a_{ik}^{\alpha\gamma} \frac{\partial \chi_j^{\gamma\beta}}{\partial y_k} \right\}. \quad (2.11)$$

In the a.p. case, solutions of the elliptic system (2.10) in \mathbb{R}^d in general do not exist in the class of a.p. functions. Although the existence of correctors is not needed for qualitative homogenization ($\psi_{kj}^{\gamma\beta}$ plays the role of $\frac{\partial \chi_j^{\gamma\beta}}{\partial y_k}$ in the definition of the homogenized coefficients), the study of the so-called approximate correctors is fundamental in understanding the quantitative properties in homogenization of \mathcal{L}_ε . As indicated in the Introduction, in this paper we will carry out a systematic study of the approximate correctors χ_T under the assumption that A is a.p. in the sense of H. Weyl.

We end this section with a few definitions and observations that will be useful to us. Let $1 \leq p < \infty$. We say $F \in L_{\text{loc,unif}}^p(\mathbb{R}^d)$ if $F \in L_{\text{loc}}^p(\mathbb{R}^d)$ and

$$\sup_{x \in \mathbb{R}^d} \int_{B(x,1)} |F|^p < \infty. \quad (2.12)$$

For $F \in L_{\text{loc,unif}}^p(\mathbb{R}^d)$ and $R > 0$, we define the norm,

$$\|F\|_{S_R^p} := \sup_{x \in \mathbb{R}^d} \left(\oint_{B(x,R)} |F|^p \right)^{1/p}. \quad (2.13)$$

Note that if $0 < r < R < \infty$, then

$$\|F\|_{S_R^p} \leq C \|F\|_{S_r^p}, \quad (2.14)$$

where C depends only on d and p . Let

$$\|F\|_{W^p} := \limsup_{R \rightarrow \infty} \|F\|_{S_R^p}. \quad (2.15)$$

It follows from (2.14) that $\|F\|_{W^p} \leq C_p \|F\|_{S_R^p}$ for any $R > 0$. A function F is said to be W^p (resp., S_R^p) a.p. if $F \in L_{\text{loc,unif}}^p(\mathbb{R}^d)$ and there exists a sequence of trigonometric polynomials $\{t_n\}$ such that $\|F - t_n\|_{W^p} \rightarrow 0$ (resp., $\|F - t_n\|_{S_R^p} \rightarrow 0$), as $n \rightarrow \infty$.

For $y, z \in \mathbb{R}^d$, define the difference operator

$$\Delta_{yz}g(x) := g(x+y) - g(x+z). \quad (2.16)$$

Proposition 2.2. *Let $g \in L_{\text{loc,unif}}^p(\mathbb{R}^d)$ for some $1 \leq p < \infty$. Then g is W^p a.p. if and only if*

$$\sup_{y \in \mathbb{R}^d} \inf_{|z| \leq L} \|\Delta_{yz}(g)\|_{S_R^p} \rightarrow 0 \quad \text{as } L, R \rightarrow \infty. \quad (2.17)$$

Proof. A subset E of \mathbb{R}^d is said to be relatively dense in \mathbb{R}^d if there exists $L > 0$ such that $\mathbb{R}^d = E + B(0, L)$; i.e., any x in \mathbb{R}^d can be written as $y + z$ for some $y \in E$ and $z \in B(0, L)$. It is known that $g \in L_{\text{loc,unif}}^p$ is W^p a.p. if and only if for any $\varepsilon > 0$, there exists $R = R_\varepsilon > 0$ such that the set

$$\left\{ \tau \in \mathbb{R}^d : \|g(\cdot + \tau) - g(\cdot)\|_{S_R^p} < \varepsilon \right\}$$

is relatively dense in \mathbb{R}^d [6]. It is not hard to see that this is equivalent to (2.17). \square

Let $P = P_k = \{(y_1, z_1), \dots, (y_k, z_k)\}$, where $(y_i, z_i) \in \mathbb{R}^d \times \mathbb{R}^d$, and

$$Q = \{(y_{i_1}, z_{i_1}), \dots, (y_{i_\ell}, z_{i_\ell})\}$$

be a subset of P with $i_1 < i_2 < \dots < i_\ell$. Define

$$\Delta_Q(g) = \Delta_{y_{i_1} z_{i_1}} \cdots \Delta_{y_{i_\ell} z_{i_\ell}}(g).$$

To quantify the almost periodicity of the coefficient matrix A , we introduce

$$\rho_k(L, R) = \sup_{y_1 \in \mathbb{R}^d} \inf_{|z_1| \leq L} \cdots \sup_{y_k \in \mathbb{R}^d} \inf_{|z_k| \leq L} \sum \|\Delta_{Q_1}(A)\|_{S_R^p} \cdots \|\Delta_{Q_\ell}(A)\|_{S_R^p}, \quad (2.18)$$

where the sum is taken over all partitions of $P = Q_1 \cup Q_2 \cup \dots \cup Q_\ell$ with $1 \leq \ell \leq k$. The exponent p in (2.18) depends on k and is given by

$$\frac{k}{p} = \frac{1}{2} - \frac{1}{\bar{q}}, \quad (2.19)$$

where $\bar{q} > 2$ is the exponent in the reverse Hölder estimate (3.4) below and depends only on d, m and μ .

3 Definition of approximate correctors

In this section and next we construct the approximate correctors and establish some preliminary estimates, under the assumptions that $A = (a_{ij}^{\alpha\beta})$ satisfies the ellipticity condition (1.2) and $A \in APW^2(\mathbb{R}^d)$. As in [17], the existence of the approximate correctors is based on the following lemma.

Lemma 3.1. *Suppose A satisfies the ellipticity condition (1.2). Let $F \in L_{\text{loc}, \text{unif}}^2(\mathbb{R}^d; \mathbb{R}^m)$ and $f \in L_{\text{loc}, \text{unif}}^2(\mathbb{R}^d; \mathbb{R}^{m \times d})$. Then, for any $T > 0$, there exists a unique $u = u_T \in H_{\text{loc}}^1(\mathbb{R}^d; \mathbb{R}^m)$ such that $u, \nabla u \in L_{\text{loc}, \text{unif}}^2(\mathbb{R}^d)$ and*

$$-\operatorname{div}(A \nabla u) + T^{-2}u = F + \operatorname{div}(f) \quad \text{in } \mathbb{R}^d. \quad (3.1)$$

Moreover, the solution u satisfies the estimate

$$\|\nabla u\|_{S_T^2} + T^{-1}\|u\|_{S_T^2} \leq C \left\{ \|g\|_{S_T^2} + T\|f\|_{S_T^2} \right\}, \quad (3.2)$$

where C depends only on d, m and μ .

Proof. See e.g. [17]. □

The weak solution of (3.1), given by Lemma 3.1, satisfies

$$\|\nabla u\|_{S_T^q} \leq C \left\{ \|f\|_{S_T^q} + T\|F\|_{S_T^2} \right\}, \quad (3.3)$$

for $2 \leq q \leq \bar{q}$, where $\bar{q} > 2$ and $C > 0$ depend only on d, m and μ . This follows from the reverse Hölder estimate [10] for weak solutions of $-\operatorname{div}(A \nabla v) = F + \operatorname{div}(f)$ in $2B = B(x_0, 2R)$,

$$\left(\int_B |\nabla v|^{\bar{q}} \right)^{1/\bar{q}} \leq C \left\{ \left(\int_{2B} |\nabla v|^2 \right)^{1/2} + R \left(\int_{2B} |F|^2 \right)^{1/2} + \left(\int_{2B} |f|^{\bar{q}} \right)^{1/\bar{q}} \right\}. \quad (3.4)$$

For $T > 0$, let $u = \chi_{T,j}^\beta = (\chi_{T,j}^{1\beta}, \dots, \chi_{T,j}^{m\beta})$ be the weak solution of

$$-\operatorname{div}(A\nabla u) + T^{-2}u = \operatorname{div}(A\nabla P_j^\beta) \quad \text{in } \mathbb{R}^d, \quad (3.5)$$

given in Lemma 3.1, where $P_j^\beta = x_j e^\beta$. The matrix-valued functions $\chi_T = (\chi_{T,j}^\beta) = (\chi_{T,j}^{\alpha\beta})$ are called the approximate correctors for \mathcal{L}_ε . It follows from (3.2) that

$$\|\nabla \chi_T\|_{S_T^2} + T^{-1}\|\chi_T\|_{S_T^2} \leq C, \quad (3.6)$$

where C depends only on d, m and μ . Also, by (3.3),

$$\|\nabla \chi_T\|_{S_T^{\bar{q}}} \leq C \quad (3.7)$$

for some $\bar{q} > 2$. Note that by Sobolev imbedding, if $d \geq 3$,

$$T^{-1}\|\chi_T\|_{S_T^p} \leq C \quad (3.8)$$

for some $p > 2d/(d-2)$. If $d = 2$, we have $\|\chi_T\|_\infty \leq CT$. Furthermore, by the De Giorgi - Nash estimates, we also have $\|\chi_T\|_\infty \leq CT$, if $m = 1$.

Lemma 3.2. *Let $u \in H_{\text{loc}}^1(\mathbb{R}^d; \mathbb{R}^m)$ be a weak solution of (3.1) in \mathbb{R}^d , given by Lemma 3.1. Then*

$$\|\nabla u\|_{S_R^2} + T^{-1}\|u\|_{S_R^2} \leq C \left\{ \|f\|_{S_R^2} + T\|F\|_{S_R^2} \right\} \quad (3.9)$$

for any $R \geq T$, where C depends on d, m and μ .

Proof. It follows by Caccioppoli's inequality that

$$\begin{aligned} & \int_{B(x_0, R)} |\nabla u|^2 + \int_{B(x_0, R)} T^{-2}|u|^2 \\ & \leq \frac{C}{R^2} \int_{B(x_0, 2R)} |u|^2 + C \int_{B(x_0, 2R)} |f|^2 + C \int_{B(x_0, 2R)} T^2|F|^2 \end{aligned} \quad (3.10)$$

for any $x_0 \in \mathbb{R}^d$ and $R > 0$, where C depends only on d, m and μ . This, together with the observation $\|F\|_{S_{2R}^2} \leq C_d \|F\|_{S_R^2}$, gives

$$\|\nabla u\|_{S_R^2} + T^{-1}\|u\|_{S_R^2} \leq C \left\{ \|f\|_{S_R^2} + T\|F\|_{S_R^2} \right\} + CR^{-1}\|u\|_{S_R^2},$$

from which the estimate (3.9) follows if $R \geq 2CT$. Finally, we observe that the case $T \leq R < 2CT$ follows directly from (3.2). \square

Lemma 3.3. *Suppose A satisfies (1.2). Then there exists some $2 < p < \infty$, depending only on d, m and μ , such that for any $y, z \in \mathbb{R}^d$,*

$$\|\Delta_{yz}(\nabla \chi_T)\|_{S_R^2} + T^{-1}\|\Delta_{yz}(\chi_T)\|_{S_R^2} \leq C \|\Delta_{yz}(A)\|_{S_R^p}, \quad (3.11)$$

where $R \geq T$ and C depends only on d, m and μ .

Proof. Fix $1 \leq j \leq d, 1 \leq \beta \leq m$, and $y, z \in \mathbb{R}^d$. Let

$$u(x) = \chi_{T,j}^\beta(x+y) - \chi_{T,j}^\beta(x+z) \quad \text{and} \quad v(x) = \chi_{T,j}^\beta(x+z).$$

Then

$$\begin{aligned} -\operatorname{div}(A(x+y)\nabla u) + T^{-2}u &= \operatorname{div}((A(x+y) - A(x+z))\nabla P_j^\beta) \\ &\quad + \operatorname{div}((A(x+y) - A(x+z))\nabla v). \end{aligned} \quad (3.12)$$

It follows from Lemma 3.2 that for any $R \geq T$,

$$\begin{aligned} &\|\nabla u\|_{S_R^2} + T^{-1}\|u\|_{S_R^2} \\ &\leq C \|\Delta_{yz}(A)\|_{S_R^2} + C \sup_{x_0 \in \mathbb{R}^d} \left(\int_{B(x_0, R)} |\Delta_{yz}(A)|^2 |\nabla v|^2 dx \right)^{1/2}. \end{aligned} \quad (3.13)$$

By (3.7) we have

$$\|\nabla v\|_{S_R^{\bar{q}}} \leq C \|\nabla v\|_{S_T^{\bar{q}}} \leq C.$$

This, together with Hölder's inequality, allows us to bound the last term in the r.h.s. of (3.13) by

$$C \sup_{x_0 \in \mathbb{R}^d} \left(\int_{B(x_0, R)} |\Delta_{yz}(A)|^p dx \right)^{1/p},$$

where $\frac{1}{p} + \frac{1}{\bar{q}} = \frac{1}{2}$ and \bar{q} is given by (3.7). In view of (3.13) we have proved the estimate (3.11). \square

Theorem 3.4. *Suppose that A satisfies (1.2) and $A \in APW^2(\mathbb{R}^d)$. Then $\chi_T, \nabla \chi_T \in APW^2(\mathbb{R}^d)$.*

Proof. By Lemma 3.3 we obtain

$$\begin{aligned} &\sup_{y \in \mathbb{R}^d} \inf_{|z| \leq L} \|\Delta_{yz}(\nabla \chi_T)\|_{S_R^2} + T^{-1} \sup_{y \in \mathbb{R}^d} \inf_{|z| \leq L} \|\Delta_{yz}(\chi_T)\|_{S_R^2} \\ &\leq C \|A\|_\infty^{1-2/p} \sup_{y \in \mathbb{R}^d} \inf_{|z| \leq L} \|\Delta_{yz}(A)\|_{S_R^2}^{2/p}, \end{aligned} \quad (3.14)$$

where $R \geq T, 0 < L < \infty$, and C depends only on d, m and μ . Since $A \in APW^2(\mathbb{R}^d)$, the r.h.s. of (3.14) goes to zero as $L, R \rightarrow \infty$. It follows that the l.h.s. of (3.14) goes to zero as $L, R \rightarrow \infty$. Since $\chi_T, \nabla \chi_T \in L_{\text{loc}, \text{unif}}^2$, by Proposition 2.2, this implies that $\chi_T, \nabla \chi_T \in APW^2(\mathbb{R}^d)$. \square

It follows from the equation (3.5), and Theorem 3.4 that if $A \in APW^2(\mathbb{R}^d)$ and $u = \chi_{T,j}^\beta$ for some $1 \leq j \leq d$ and $1 \leq \beta \leq m$, then

$$\left\langle a_{ik}^{\alpha\gamma} \frac{\partial u^\gamma}{\partial x_k} \frac{\partial v^\alpha}{\partial x_i} \right\rangle + T^{-2} \langle u^\alpha v^\alpha \rangle = - \left\langle a_{ij}^{\alpha\beta} \frac{\partial v^\alpha}{\partial x_i} \right\rangle, \quad (3.15)$$

for any $v = (v^\alpha) \in H_{\text{loc}}^1(\mathbb{R}^d; \mathbb{R}^m)$ such that $v^\alpha, \nabla v^\alpha \in B^2(\mathbb{R}^d)$. This implies that

$$\frac{\partial}{\partial x_i} \left(\chi_{T,j}^{\alpha\beta} \right) \rightarrow \psi_{ij}^{\alpha\beta} \quad \text{strongly in } B^2(\mathbb{R}^d) \text{ as } T \rightarrow \infty, \quad (3.16)$$

where $\psi = (\psi_{ij}^{\alpha\beta})$ is defined by (2.4), and that

$$T^{-2}\langle |\chi_T|^2 \rangle \rightarrow 0 \quad \text{as } T \rightarrow \infty. \quad (3.17)$$

In fact, it was observed in [17],

$$\mu\langle |\psi - \nabla\chi_T|^2 \rangle + T^{-2}\langle |\chi_T|^2 \rangle \leq \left\langle a_{ik}^{\alpha\gamma} \left[\psi_{kj}^{\gamma\beta} - \frac{\partial}{\partial x_k} (\chi_{T,j}^{\gamma\beta}) \right] \psi_{ij}^{\alpha\beta} \right\rangle. \quad (3.18)$$

4 Estimates of approximate correctors, part I

The goal of this section is to establish the following.

Theorem 4.1. *Suppose that $A \in APW^2(\mathbb{R}^d)$ and satisfies the ellipticity condition (1.2). Then*

$$T^{-1}\|\chi_T\|_{S_T^2} \leq C \inf_{0 < L < T} \left\{ \rho_1(L, T) + \frac{L}{T} \right\}, \quad (4.1)$$

where $\rho_1(L, T)$ is defined by (2.18) and C depends only on d , m and μ .

The estimate (4.1) follows from a general inequality (4.2), which may be of independent interest. The inequality allows us to bound the local (uniform) L^2 norm of a function at a specific scale by its oscillation and gradient.

We use $Q(x, R)$ to denote the (closed) cube centered at x with side length R .

Theorem 4.2. *Let $u \in H_{\text{loc}}^1(\mathbb{R}^d)$. Suppose that $u, \nabla u \in L_{\text{loc}, \text{unif}}^2(\mathbb{R}^d)$ and*

$$M = \lim_{r \rightarrow \infty} \oint_{Q(0, r)} u \quad \text{exists.}$$

Then there exists $C > 0$, depending only on d , such that for any $0 < L \leq R < \infty$,

$$\|u\|_{S_R^2} \leq |M| + C \left\{ \sup_{y \in \mathbb{R}^d} \inf_{|z| \leq L} \|\Delta_{yz}(u)\|_{S_R^2} + L \|\nabla u\|_{S_R^2} \right\}. \quad (4.2)$$

The proof of Theorem 4.2 relies on the following two lemmas.

Lemma 4.3. *Let $u \in H_{\text{loc}}^1(\mathbb{R}^d)$. Suppose that $u, \nabla u \in L_{\text{loc}, \text{unif}}^2(\mathbb{R}^d)$. Then, for any $0 < L, R < \infty$,*

$$\begin{aligned} & \sup_{x \in \mathbb{R}^d} \left(\oint_{Q(x, R)} |u|^2 \right)^{1/2} \\ & \leq C \sup_{y \in \mathbb{R}^d} \inf_{|z| \leq L} \|\Delta_{yz}(u)\|_{S_R^2} + CL \|\nabla u\|_{S_R^2} + \sup_{x \in \mathbb{R}^d} \left| \oint_{Q(x, R)} u \right|, \end{aligned} \quad (4.3)$$

where C depends only on d .

Proof. Note that for any $x \in \mathbb{R}^d$,

$$\begin{aligned}
\left(\int_{Q(x,R)} |u|^2 \right)^{1/2} &\leq \left(\int_{Q(x,R)} \left| u(t) - \int_{Q(x,R)} u(t+y) dy \right|^2 dt \right)^{1/2} + \sup_{z \in \mathbb{R}^d} \left| \int_{Q(z,R)} u \right| \\
&= \left(\int_{Q(x,R)} \left| \int_{Q(x,R)} [u(t) - u(t+y)] dy \right|^2 dt \right)^{1/2} + \sup_{z \in \mathbb{R}^d} \left| \int_{Q(z,R)} u \right| \\
&\leq \int_{Q(x,R)} \left(\int_{Q(x,R)} |u(t) - u(t+y)|^2 dt \right)^{1/2} dy + \sup_{z \in \mathbb{R}^d} \left| \int_{Q(z,R)} u \right| \\
&\leq C \sup_{y \in \mathbb{R}^d} \|u(\cdot) - u(\cdot + y)\|_{S_R^2} + \sup_{z \in \mathbb{R}^d} \left| \int_{Q(z,R)} u \right|,
\end{aligned}$$

where we have used Minkowski's inequality for the second inequality.

Next, using

$$\begin{aligned}
\|u(\cdot) - u(\cdot + y)\|_{S_R^2} &\leq \|u(\cdot + y) - u(\cdot + z)\|_{S_R^2} + \|u(\cdot + z) - u(\cdot)\|_{S_R^2} \\
&\leq \|u(\cdot + y) - u(\cdot + z)\|_{S_R^2} + L \|\nabla u\|_{S_R^2},
\end{aligned}$$

where $z \in \mathbb{R}^d$ and $|z| \leq L$, we obtain

$$\sup_{y \in \mathbb{R}^d} \|u(\cdot) - u(\cdot + y)\|_{S_R^2} \leq \sup_{y \in \mathbb{R}^d} \inf_{|z| \leq L} \|\Delta_{yz}(u)\|_{S_R^2} + L \|\nabla u\|_{S_R^2}.$$

This completes the proof. \square

Lemma 4.4. *Let $u \in L_{\text{loc}, \text{unif}}^p(\mathbb{R}^d)$ for some $p > 1$. Suppose that*

$$M = \lim_{r \rightarrow \infty} \int_{Q(0,r)} u \text{ exists.}$$

Then, for any $0 < L \leq R < \infty$,

$$\begin{aligned}
\sup_{x \in \mathbb{R}^d} \left| \int_{Q(x,R)} u - M \right| &\leq 2 \sup_{y \in \mathbb{R}^d} \inf_{\substack{z \in \mathbb{R}^d \\ |z| \leq L}} \int_{Q(0,R)} |u(t+y) - u(t+z)| dt \\
&\quad + C \left(\frac{L}{R} \right)^{1/p'} \sup_{x \in \mathbb{R}^d} \left(\int_{Q(x,R)} |u|^p \right)^{1/p},
\end{aligned} \tag{4.4}$$

where C depends only on d and p .

Proof. Observe that if $0 < L \leq R$ and $|z| \leq L$,

$$\begin{aligned}
\left| \int_{Q(z,R)} u - \int_{Q(0,R)} u \right| &\leq \frac{1}{R^d} \int_{Q(z,R) \setminus Q(0,R)} |u| + \frac{1}{R^d} \int_{Q(0,R) \setminus Q(z,R)} |u| \\
&\leq C \left(\frac{L}{R} \right)^{1/p'} \sup_{x \in \mathbb{R}^d} \left(\int_{Q(x,R)} |u|^p \right)^{1/p},
\end{aligned} \tag{4.5}$$

where C depends only on d and p . It follows that for any $y \in \mathbb{R}^d$,

$$\begin{aligned} \left| \int_{Q(y,R)} u - \int_{Q(0,R)} u \right| &\leq \left| \int_{Q(y,R)} u - \int_{Q(z,R)} u \right| + \left| \int_{Q(z,R)} u - \int_{Q(0,R)} u \right| \\ &\leq \left| \int_{Q(y,R)} u - \int_{Q(z,R)} u \right| + C \left(\frac{L}{R} \right)^{1/p'} \sup_{x \in \mathbb{R}^d} \left(\int_{Q(x,R)} |u|^p \right)^{1/p}. \end{aligned}$$

Hence,

$$\begin{aligned} &\sup_{y \in \mathbb{R}^d} \left| \int_{Q(y,R)} u - \int_{Q(0,R)} u \right| \\ &\leq \sup_{y \in \mathbb{R}^d} \inf_{\substack{z \in \mathbb{R}^d \\ |z| \leq L}} \int_{Q(0,R)} |u(t+y) - u(t+z)| dt + C \left(\frac{L}{R} \right)^{1/p'} \sup_{x \in \mathbb{R}^d} \left(\int_{Q(x,R)} |u|^p \right)^{1/p}. \end{aligned}$$

Using

$$\left| \int_{Q(0,R)} u - \int_{Q(0,kR)} u \right| \leq \frac{1}{k^d} \sum_{i=1}^{k^d} \left| \int_{Q(0,R)} u - \int_{Q(x_i,R)} u \right|,$$

where $k \geq 1$ and $Q(0, kR) = \cup_{i=1}^{k^d} Q(x_i, R)$, we then obtain

$$\begin{aligned} &\sup_{y \in \mathbb{R}^d} \left| \int_{Q(y,R)} u - \int_{Q(0,kR)} u \right| \\ &\leq \sup_{y \in \mathbb{R}^d} \left| \int_{Q(y,R)} u - \int_{Q(0,R)} u \right| + \left| \int_{Q(0,R)} u - \int_{Q(0,kR)} u \right| \\ &\leq 2 \sup_{y \in \mathbb{R}^d} \left| \int_{Q(y,R)} u - \int_{Q(0,R)} u \right| \\ &\leq 2 \sup_{y \in \mathbb{R}^d} \inf_{\substack{z \in \mathbb{R}^d \\ |z| \leq L}} \int_{Q(0,R)} |u(t+y) - u(t+z)| dt + C \left(\frac{L}{R} \right)^{1/p'} \sup_{x \in \mathbb{R}^d} \left(\int_{Q(x,R)} |u|^p \right)^{1/p}, \end{aligned}$$

from which the lemma follows by letting $k \rightarrow \infty$. \square

Remark 4.5. In the place of (4.5) we may also use

$$\begin{aligned} \left| \int_{Q(z,R)} u - \int_{Q(0,R)} u \right| &\leq \left| \int_{Q(z,R)} u - \int_{Q(0,2R)} u \right| + \left| \int_{Q(0,R)} u - \int_{Q(0,2R)} u \right| \\ &\leq CR \left(\int_{Q(0,2R)} |\nabla u|^2 \right)^{1/2}, \end{aligned}$$

where the last step follows by Poincaré inequality. This would lead to the estimate

$$\begin{aligned} \sup_{x \in \mathbb{R}^d} \left| \int_{Q(x,R)} u - M \right| &\leq 2 \sup_{y \in \mathbb{R}^d} \inf_{\substack{z \in \mathbb{R}^d \\ |z| \leq R}} \int_{Q(0,R)} |u(t+y) - u(t+z)| dt \\ &\quad + CR \sup_{x \in \mathbb{R}^d} \left(\int_{Q(x,R)} |\nabla u|^2 \right)^{1/2} \end{aligned} \tag{4.6}$$

for any $0 < R < \infty$.

Remark 4.6. It follows from Lemma 4.4 that functions in $APW^2(\mathbb{R}^d)$ have uniform mean values. In particular,

$$\sup_{x \in \mathbb{R}^d} \left| \int_{Q(x,R)} A - \langle A \rangle \right| \leq C \inf_{0 < L < R} \left\{ \rho_1(L, R) + \frac{L}{R} \right\}, \quad (4.7)$$

where C depends only on d, m and μ .

Remark 4.7. Let $u(x) = A(x) \nabla \chi_T(x)$, $R = T$ and $p = 2$ in Lemma 4.4. Observe that by (3.11),

$$\begin{aligned} & \int_{Q(0,T)} |u(t+y) - u(t+z)| dt \\ & \leq C \left(\int_{Q(0,T)} |A(t+y) - A(t+z)|^2 dt \right)^{1/2} + C \int_{Q(0,T)} |\nabla \chi_T(t+y) - \nabla \chi_T(t+z)| dt \\ & \leq C \sup_{x \in \mathbb{R}^d} \left(\int_{Q(x,T)} |A(t+y) - A(t+z)|^q dt \right)^{1/q}, \end{aligned}$$

for some $q \in (2, \infty)$. It follows by Lemma 4.4 that

$$\sup_{x \in \mathbb{R}^d} \left| \int_{Q(x,T)} A \nabla \chi_T - \langle A \nabla \chi_T \rangle \right| \leq C \inf_{0 < L < T} \left\{ \rho_1(L, T) + \left(\frac{L}{T} \right)^{1/2} \right\}, \quad (4.8)$$

where C depends only on d, m and μ . This estimate will be used in the proof of Theorem 5.1.

We are now in a position to give the proof of Theorem 4.2.

Proof of Theorem 4.2. By considering the function $u - M$ we may assume that $\langle u \rangle = 0$. It follows from Lemmas 4.3 and 4.4 that for any $0 < L \leq R < \infty$,

$$\begin{aligned} \|u\|_{S_R^2} & \leq C \sup_{y \in \mathbb{R}^d} \inf_{|z| \leq L} \|\Delta_{yz}(u)\|_{S_R^2} + CL \|\nabla u\|_{S_R^2} + \sup_{x \in \mathbb{R}^d} \left| \int_{Q(x,R)} u \right| \\ & \leq C \sup_{y \in \mathbb{R}^d} \inf_{|z| \leq L} \|\Delta_{yz}(u)\|_{S_R^2} + CL \|\nabla u\|_{S_R^2} + C \left(\frac{L}{R} \right)^{1/2} \|u\|_{S_R^2}, \end{aligned}$$

where C depends only on d . Since $u, \nabla u \in L_{\text{loc}, \text{unif}}^2$, it follows that $\|u\|_{S_R^2}$ and $\|\nabla u\|_{S_R^2}$ are finite for any $R > 0$. Next, we fix a constant $\theta \in (0, 1)$ so small that $C\theta^{1/2} < 1/2$. Then, if $0 < L \leq \theta R$, we have

$$\|u\|_{S_R^2} \leq C \sup_{y \in \mathbb{R}^d} \inf_{|z| \leq L} \|\Delta_{yz}(u)\|_{S_R^2} + CL \|\nabla u\|_{S_R^2}.$$

Finally, we observe that if $\theta R < L \leq R$, the estimate (4.2) follows from Lemma 4.3 and (4.6). \square

Proof of Theorem 4.1. To see (4.1), we let $L = T$ in (4.2) and use (3.11) and the fact that $\|\nabla \chi_T\|_{S_T^2} \leq C$. \square

5 A compactness theorem

In this section we establish a compactness theorem, which extends Theorem 2.1 in the case $A \in APW^2(\mathbb{R}^d)$. It will play a key role in the compactness argument in the next section. Throughout this section we will assume that $A \in APW^2(\mathbb{R}^d)$ and satisfies the ellipticity condition (1.2).

Theorem 5.1. *Let Ω be a bounded Lipschitz domain in \mathbb{R}^d . Suppose that $\{u_\ell\}_{\ell=1}^\infty \subset H^1(\Omega; \mathbb{R}^m)$ and*

$$-\operatorname{div}(A^\ell(x/\varepsilon_\ell)\nabla u_\ell) + \lambda_\ell u_\ell = F_\ell \quad \text{in } \Omega,$$

where $\varepsilon_\ell \rightarrow 0$, $\lambda_\ell \rightarrow \lambda$, and $A^\ell(y) = A(y + x_\ell)$ for some $x_\ell \in \mathbb{R}^d$. Assume that $u_\ell \rightharpoonup u_0$ weakly in $H^1(\Omega; \mathbb{R}^m)$ and $F_\ell \rightarrow F_0$ strongly in $H^{-1}(\Omega; \mathbb{R}^m)$. Then $A^\ell(x/\varepsilon_\ell)\nabla u_\ell \rightharpoonup \widehat{A}\nabla u_0$ weakly in $L^2(\Omega; \mathbb{R}^{m \times d})$, and $u_0 \in H^1(\Omega; \mathbb{R}^m)$ is a weak solution of

$$-\operatorname{div}(\widehat{A}\nabla u_0) + \lambda u_0 = F_0 \quad \text{in } \Omega,$$

where \widehat{A} is the homogenized matrix of A .

We will prove Theorem 5.1 by using Tartar's method of test functions and the following lemma.

Lemma 5.2. *Let $\{x_\ell\} \subset \mathbb{R}^d$, $\varepsilon_\ell \rightarrow 0$, and $T_\ell = \varepsilon_\ell^{-1}$. Then for any bounded domain Ω ,*

$$\varepsilon_\ell \chi_{T_\ell}((x/\varepsilon_\ell) + x_\ell) \rightharpoonup 0 \quad \text{weakly in } H^1(\Omega), \quad (5.1)$$

and

$$a_{ik}^{\alpha\gamma}((x/\varepsilon_\ell) + x_\ell) \frac{\partial}{\partial x_k} \left\{ x_j \delta^{\gamma\beta} + \varepsilon_\ell \chi_{T_\ell, j}^{\gamma\beta}((x/\varepsilon_\ell) + x_\ell) \right\} \rightharpoonup \widehat{a}_{ij}^{\alpha\beta} \quad \text{weakly in } L^2(\Omega), \quad (5.2)$$

as $\ell \rightarrow \infty$.

Proof. We start with the proof of (5.2). Let $\{g_\ell\}$ denote the sequence in (5.2). Since $T_\ell = \varepsilon_\ell^{-1}$, in view of estimate (3.7), $\{g_\ell\}$ is bounded in $L^2(\Omega)$. Thus, by a density argument, it suffices to show that

$$\int_{B(z, r)} g_\ell \rightarrow \widehat{a}_{ij}^{\alpha\beta} \quad \text{as } \ell \rightarrow \infty, \quad (5.3)$$

for any ball $B(z, r)$ in \mathbb{R}^d . Recall that $\widehat{A} = \langle A \rangle + \langle A\psi \rangle$, where $\psi = (\psi_{ij}^{\alpha\beta})$ is defined by (2.4). Hence,

$$\begin{aligned} \left| \int_{B(z, r)} g_\ell - \widehat{a}_{ij}^{\alpha\beta} \right| &= \left| \int_{B(x_\ell + T_\ell z, r T_\ell)} a_{ik}^{\alpha\gamma}(y) \left\{ \delta_{jk} \delta^{\gamma\beta} + \frac{\partial}{\partial y_k} (\chi_{T_\ell, j}^{\gamma\beta}) \right\} dy - \widehat{a}_{ij}^{\alpha\beta} \right| \\ &\leq \sup_{x \in \mathbb{R}^d} \left| \int_{B(x, r T_\ell)} A - \langle A \rangle \right| + \sup_{x \in \mathbb{R}^d} \left| \int_{B(x, r T_\ell)} A \nabla \chi_{T_\ell} - \langle A \nabla \chi_{T_\ell} \rangle \right| + |\langle A(\nabla \chi_{T_\ell} - \psi) \rangle|. \end{aligned}$$

This, together with Remark 4.6, Remark 4.7 and (3.16), gives (5.2).

Finally, let $\{f_\ell\}$ denote the sequence in (5.1). It follows from estimates (3.8) and (3.7) that $\{f_\ell\}$ is bounded in $H^1(\Omega)$. Also, by the estimate (4.1), $f_\ell \rightarrow 0$ strongly in $L^2(\Omega)$. This implies that $f_\ell \rightharpoonup 0$ weakly in $H^1(\Omega)$. \square

Proof of Theorem 5.1. Let $p_\ell(x) = A^\ell(x/\varepsilon_\ell)\nabla u_\ell(x)$. Observe that since $\{u_\ell\}$ is bounded in $H^1(\Omega; \mathbb{R}^m)$, $\{p_\ell\}$ is bounded in $L^2(\Omega; \mathbb{R}^{m \times d})$. We will show that if a subsequence $\{p_{\ell'}\}$ converges weakly in $L^2(\Omega; \mathbb{R}^{m \times d})$ to p_0 , then $p_0 = \widehat{A}\nabla u_0$. This would imply that the full sequence converges weakly to $\widehat{A}\nabla u_0$. As a result, we also obtain $-\operatorname{div}(\widehat{A}\nabla u_0) + \lambda u_0 = F_0$ in Ω .

Without loss of generality let us assume that p_ℓ converges weakly in $L^2(\Omega; \mathbb{R}^{m \times d})$ to p_0 , as $\ell \rightarrow \infty$. Let $\psi \in C_0^1(\Omega)$. Fix $1 \leq j \leq d$ and $1 \leq \beta \leq m$. Let $T_\ell = \varepsilon_\ell^{-1}$. Note that

$$\begin{aligned}
& \langle F_\ell, (P_j^\beta + \varepsilon_\ell \chi_{T_\ell, j}^{\ell* \beta}(x/\varepsilon_\ell))\psi \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \\
&= \lambda_\ell \int_\Omega u_\ell (P_j^\beta + \varepsilon_\ell \chi_{T_\ell, j}^{\ell* \beta}(x/\varepsilon_\ell))\psi dx \\
&\quad + \int_\Omega A^\ell(x/\varepsilon_\ell)\nabla u_\ell \cdot \nabla \left\{ (P_j^\beta + \varepsilon_\ell \chi_{T_\ell, j}^{\ell* \beta}(x/\varepsilon_\ell))\psi \right\} dx \\
&= \lambda_\ell \int_\Omega u_\ell (P_j^\beta + \varepsilon_\ell \chi_{T_\ell, j}^{\ell* \beta}(x/\varepsilon_\ell))\psi dx \\
&\quad + \int_\Omega A^\ell(x/\varepsilon_\ell)\nabla u_\ell \cdot \nabla (P_j^\beta + \varepsilon_\ell \chi_{T_\ell, j}^{\ell* \beta}(x/\varepsilon_\ell))\psi dx \\
&\quad + \int_\Omega A^\ell(x/\varepsilon_\ell)\nabla u_\ell \cdot (P_j^\beta + \varepsilon_\ell \chi_{T_\ell, j}^{\ell* \beta}(x/\varepsilon_\ell))(\nabla \psi) dx \\
&= \lambda_\ell \int_\Omega u_\ell (P_j^\beta + \varepsilon_\ell \chi_{T_\ell, j}^{\ell* \beta}(x/\varepsilon_\ell))\psi dx - \int_\Omega u_\ell \cdot \varepsilon_\ell \chi_{T_\ell, j}^{\ell* \beta}(x/\varepsilon_\ell)\psi dx \\
&\quad - \int_\Omega u_\ell \cdot (A^\ell)^*(x/\varepsilon_\ell)\nabla (P_j^\beta + \varepsilon_\ell \chi_{T_\ell, j}^{\ell* \beta}(x/\varepsilon_\ell))(\nabla \psi) dx \\
&\quad + \int_\Omega A^\ell(x/\varepsilon_\ell)\nabla u_\ell \cdot (P_j^\beta + \varepsilon_\ell \chi_{T_\ell, j}^{\ell* \beta}(x/\varepsilon_\ell))(\nabla \psi) dx,
\end{aligned} \tag{5.4}$$

where $\chi_{T_\ell, j}^{\ell* \beta}$ denote the approximate correctors for the adjoint matrix $(A^\ell)^*$. We point out that the following equation

$$\operatorname{div} \left\{ (A^\ell)^*(x/\varepsilon_\ell)\nabla (P_j^\beta + \varepsilon_\ell \chi_{T_\ell, j}^{\ell* \beta}(x/\varepsilon_\ell)) \right\} = -\varepsilon_\ell \chi_{T_\ell, j}^{\ell* \beta}(x/\varepsilon_\ell) \quad \text{in } \mathbb{R}^d,$$

was used for the last equality in (5.4). Since $A^\ell(y) = A(y + x_\ell)$, we have

$$\chi_{T_\ell, j}^{\ell* \beta}(y) = \chi_{T_\ell, j}^{* \beta}(y + x_\ell), \tag{5.5}$$

where $\chi_{T_\ell, j}^{* \beta}$ denote the approximate correctors for A^* .

We now let $\ell \rightarrow \infty$ in (5.4) and use Lemma 5.2 (with A^* in the place of A) to find the limit on each side. Since $F_\ell \rightarrow F_0$ strongly in $H^{-1}(\Omega; \mathbb{R}^m)$, by (5.1), the l.h.s. of (5.4) converges to $\langle F_0, P_j^\beta \psi \rangle$. Since $u_\ell \rightarrow u_0$ strongly in $L^2(\Omega; \mathbb{R}^m)$, it also follows from (5.1) that the sum of first two terms in the r.h.s. of (5.4) converges to $\lambda \int_\Omega u_0 P_j^\beta \psi dx$, and the fourth term converges to

$$\int_\Omega p_0 \cdot P_j^\beta (\nabla \psi) dx.$$

Similarly, using (5.2) and the fact that $u_\ell \rightarrow u_0$ strongly in $L^2(\Omega; \mathbb{R}^m)$, we see that the third term in the r.h.s. of (5.4) converges to

$$- \int_\Omega u_0^\alpha \cdot \widehat{a}_{ij}^{\alpha \beta} \frac{\partial \psi}{\partial x_i} dx = \int_\Omega \widehat{a}_{ij}^{\alpha \beta} \frac{\partial u_0^\alpha}{\partial x_i} \psi dx = \int_\Omega \widehat{a}_{ji}^{\beta \alpha} \frac{\partial u_0^\alpha}{\partial x_i} \psi dx,$$

where $\widehat{A}^* = (\widehat{a}_{ij}^{\alpha\beta})$ and we have used the fact $(\widehat{A})^* = \widehat{A}^*$ for the last step. As a result, we have proved that

$$\langle F_0, P_j^\beta \psi \rangle = \lambda \int_{\Omega} u_0 \cdot P_j^\beta \psi \, dx + \int_{\Omega} \widehat{a}_{ji}^{\beta\alpha} \frac{\partial u_0^\alpha}{\partial x_i} \psi \, dx + \int_{\Omega} p_0 \cdot P_j^\beta (\nabla \psi) \, dx. \quad (5.6)$$

Finally, by taking limits in the equation

$$\langle F_\ell, P_j^\beta \psi \rangle = \lambda_\ell \int_{\Omega} u_\ell \cdot P_j^\beta \psi \, dx + \int_{\Omega} p_\ell \cdot \nabla (P_j^\beta \psi) \, dx,$$

we obtain

$$\begin{aligned} \langle F_0, P_j^\beta \psi \rangle &= \lambda \int_{\Omega} u_0 \cdot P_j^\beta \psi \, dx + \int_{\Omega} p_0 \cdot \nabla (P_j^\beta \psi) \, dx \\ &= \lambda \int_{\Omega} u_0 \cdot P_j^\beta \psi \, dx + \int_{\Omega} p_0 \cdot \nabla (P_j^\beta \psi) \, dx + \int_{\Omega} p_0 \cdot P_j^\beta (\nabla \psi) \, dx. \end{aligned}$$

Since $\psi \in C_0^1(\Omega)$ is arbitrary, this, together with (5.6), implies that $p_0 \cdot (\nabla P_j^\beta) = \widehat{a}_{ji}^{\beta\alpha} \frac{\partial u_0^\alpha}{\partial x_i}$, i.e., $p_0 = \widehat{A} \nabla u_0$. The proof is complete. \square

6 Hölder estimates at large scale

In this section we establish an L^2 -based Hölder estimate at large scale for the elliptic system,

$$-\operatorname{div}(A(x/\varepsilon) \nabla u_\varepsilon) + \lambda u_\varepsilon = F + \operatorname{div}(f). \quad (6.1)$$

Throughout this section we will assume that $A \in APW^2(\mathbb{R}^d)$ and satisfies the ellipticity condition (1.2).

Theorem 6.1. *Fix $\sigma \in (0, 1)$ and $B = B(x_0, R)$ for some $x_0 \in \mathbb{R}^d$. Let $u_\varepsilon \in H^1(B; \mathbb{R}^m)$ be a weak solution of*

$$-\operatorname{div}(A(x/\varepsilon) \nabla u_\varepsilon) + \lambda u_\varepsilon = F + \operatorname{div}(f) \quad \text{in } B, \quad (6.2)$$

for $0 < \varepsilon < R$ and $\lambda \in [0, R^{-2}]$. Then, if $\varepsilon \leq r \leq R/2$,

$$\begin{aligned} & \left(\int_{B(x_0, r)} |\nabla u_\varepsilon|^2 \right)^{1/2} + \sqrt{\lambda} \left(\int_{B(x_0, r)} |u_\varepsilon|^2 \right)^{1/2} \\ & \leq C_\sigma \left(\frac{R}{r} \right)^\sigma \left\{ \frac{1}{R} \left(\int_{B(x_0, R)} |u_\varepsilon|^2 \right)^{1/2} + \sup_{\substack{x \in B(x_0, R/2) \\ r \leq t \leq R/2}} t \left(\int_{B(x, t)} |F|^2 \right)^{1/2} \right. \\ & \quad \left. + \sup_{x \in B(x_0, R/2)} \left(\int_{B(x, r)} |f|^2 \right)^{1/2} \right\}, \end{aligned} \quad (6.3)$$

where C_σ depends only on σ and A .

Remark 6.2. One may regard the estimate (6.3) as a Hölder estimate at large scale, as the estimate

$$\sup_{\substack{0 < r < R/2 \\ x \in B(x_0, R/2)}} r^\sigma \left(\int_{B(x, r)} |\nabla u_\varepsilon|^2 \right)^{1/2} < \infty$$

would imply that $u_\varepsilon \in C^{1-\sigma}(B(x_0, R/2))$. Note that since no smoothness condition is imposed on A , estimate (6.3) may fail to hold for $0 < r < \varepsilon$.

As a corollary of Theorem 6.1, we obtain a Liouville property for the elliptic operator \mathcal{L}_1 .

Corollary 6.3. *Suppose that $A \in APW^2(\mathbb{R}^d)$ and satisfies the ellipticity condition (1.2). Let $u \in H_{\text{loc}}^1(\mathbb{R}^d; \mathbb{R}^m)$ be a weak solution of $\text{div}(A\nabla u) = 0$ in \mathbb{R}^d . Assume that there exist constants $\sigma \in (0, 1)$ and $C_u > 0$ such that*

$$\left(\int_{B(0, R)} |u|^2 \right)^{1/2} \leq C_u R^\sigma \quad (6.4)$$

for all $R \geq 1$. Then u is constant in \mathbb{R}^d .

Proof. Choose $\sigma_1 \in (\sigma, 1)$. It follows from Theorem 6.1 that

$$\begin{aligned} \left(\int_{B(0, r)} |\nabla u|^2 \right)^{1/2} &\leq C \left(\frac{R}{r} \right)^{1-\sigma_1} \frac{1}{R} \left(\int_{B(0, R)} |u|^2 \right)^{1/2} \\ &\leq \frac{CR^{\sigma-\sigma_1}}{r^{1-\sigma_1}} \end{aligned}$$

for any $1 \leq r \leq R/2$. By letting $R \rightarrow \infty$, we obtain $\nabla u = 0$ in $B(0, r)$ for any $r > 1$. Thus u is constant in \mathbb{R}^d . \square

Theorem 6.1 will be proved by using a compactness argument introduced by Avellaneda and Lin [5] to the study of uniform regularity estimates in homogenization.

We begin with a Caccioppoli's inequality.

Lemma 6.4. *Let u be a weak solution of $-\text{div}(A\nabla u) + \lambda u = F + \text{div}(f)$ in $B(x_0, 2R)$ for some $x_0 \in \mathbb{R}^d$, $R > 0$, and $\lambda \geq 0$. Then*

$$\begin{aligned} \left(\int_B |\nabla u|^2 \right)^{1/2} &\leq \frac{C}{R} \left(\int_{2B} |u - \int_{2B} u|^2 \right)^{1/2} + C \left(\int_{2B} |f|^2 \right)^{1/2} \\ &\quad + CR \left\{ \lambda \left(\int_{2B} |u|^2 \right)^{1/2} + \left(\int_{2B} |F|^2 \right)^{1/2} \right\}, \end{aligned} \quad (6.5)$$

where $B = B(x_0, R)$ and C depends only on d , m and μ .

Proof. This is well known. \square

To assure that our estimates are translation invariant in the compactness argument, we introduce the set of all matrices obtained from A by translation,

$$\mathcal{A} = \left\{ M = M(y) : M(y) = A(y + z) \text{ for some } z \in \mathbb{R}^d \right\}.$$

Lemma 6.5. Fix $\sigma \in (0, 1)$. There exist $\varepsilon_0 \in (0, 1/2)$ and $\theta \in (0, 1/8)$, depending at most on σ and A , such that

$$\begin{aligned} & \left(\int_{B(0,\theta)} |u_\varepsilon - \int_{B(0,\theta)} u_\varepsilon|^2 \right)^{1/2} + \theta \sqrt{\lambda} \left(\int_{B(0,\theta)} |u_\varepsilon|^2 \right)^{1/2} \\ & \leq \theta^\sigma \left\{ \left(\int_{B(0,1)} |u_\varepsilon - \int_{B(0,1)} u_\varepsilon|^2 \right)^{1/2} + \sqrt{\lambda} \left(\int_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2} \right. \\ & \quad \left. + \varepsilon_0^{-1} \left(\int_{B(0,1)} |F|^2 \right)^{1/2} + \varepsilon_0^{-1} \left(\int_{B(0,1)} |f|^2 \right)^{1/2} \right\}, \end{aligned} \quad (6.6)$$

whenever $0 < \varepsilon < \varepsilon_0$ and $u_\varepsilon \in H^1(B(0, 1); \mathbb{R}^m)$ is a weak solution of

$$-\operatorname{div}(M(x/\varepsilon)\nabla u_\varepsilon) + \lambda u_\varepsilon = F + \operatorname{div}(f) \quad \text{in } B(0, 1) \quad (6.7)$$

for some $M \in \mathcal{A}$ and $\lambda \in [0, \varepsilon_0^2]$.

Proof. The lemma is proved by contradiction, along a line of argument used in [5] for periodic coefficients. We will show that there exist $\varepsilon_0 \in (0, 1/2)$ and $\theta \in (0, 1/8)$, depending at most on σ and A , such that whenever $0 < \varepsilon < \varepsilon_0$ and u_ε is a solution of (6.7), then

$$\begin{aligned} & \left(\int_{B(0,\theta)} |u_\varepsilon - \int_{B(0,\theta)} u_\varepsilon|^2 \right)^{1/2} + \theta \left(\int_{B(0,\theta)} |u_\varepsilon|^2 \right)^{1/2} \\ & \leq \frac{\theta^\sigma}{2} \left\{ \left(\int_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2} + \varepsilon_0^{-1} \left(\int_{B(0,1)} |F|^2 \right)^{1/2} + \varepsilon_0^{-1} \left(\int_{B(0,1)} |f|^2 \right)^{1/2} \right\}. \end{aligned} \quad (6.8)$$

We claim that (6.8) implies (6.6). In fact, assume (6.8) is true, we set $E = \int_{B(0,1)} u_\varepsilon$ and apply (6.8) to $v_\varepsilon = u_\varepsilon - E$, which is a solution of (6.7) with the r.h.s. F replaced by $F - \lambda E$. As a result, it follows that

$$\begin{aligned} & \left(\int_{B(0,\theta)} |u_\varepsilon - \int_{B(0,\theta)} u_\varepsilon|^2 \right)^{1/2} + \theta \left(\int_{B(0,\theta)} |u_\varepsilon - \int_{B(0,1)} u_\varepsilon|^2 \right)^{1/2} \\ & \leq \frac{\theta^\sigma}{2} \left\{ \left(\int_{B(0,1)} |u_\varepsilon - \int_{B(0,1)} u_\varepsilon|^2 \right)^{1/2} \right. \\ & \quad \left. + \varepsilon_0^{-1} \left(\int_{B(0,1)} |F - \lambda \int_{B(0,1)} u_\varepsilon|^2 \right)^{1/2} + \varepsilon_0^{-1} \left(\int_{B(0,1)} |f|^2 \right)^{1/2} \right\} \\ & \leq \frac{\theta^\sigma}{2} \left\{ \left(\int_{B(0,1)} |u_\varepsilon - \int_{B(0,1)} u_\varepsilon|^2 \right)^{1/2} + \sqrt{\lambda} \left(\int_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2} \right. \\ & \quad \left. + \varepsilon_0^{-1} \left(\int_{B(0,1)} |F|^2 \right)^{1/2} + \varepsilon_0^{-1} \left(\int_{B(0,1)} |f|^2 \right)^{1/2} \right\}, \end{aligned} \quad (6.9)$$

where we have used the assumption $\lambda \leq \varepsilon_0^2$ in the last step. Using the fact that $\theta, \lambda \in [0, 1]$ as well as the triangle inequality, we have

$$\theta \left(\int_{B(0,\theta)} |u_\varepsilon - \int_{B(0,1)} u_\varepsilon|^2 \right)^{1/2} \geq \theta \sqrt{\lambda} \left(\int_{B(0,\theta)} |u_\varepsilon|^2 \right)^{1/2} - \frac{\theta^\sigma}{2} \sqrt{\lambda} \left(\int_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2},$$

where we also assume that θ is small enough so that $\theta \leq \theta^\sigma/2$. This, together with (6.9), gives the desired estimate (6.6).

It remains to prove (6.8). By normalizing the r.h.s. of (6.8), without loss of generality, it suffices to show that there exist $\varepsilon_0 \in (0, 1/2)$ and $\theta \in (0, 1/8)$ so that if

$$\left(\int_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2} + \varepsilon_0^{-1} \left(\int_{B(0,1)} |F|^2 \right)^{1/2} + \varepsilon_0^{-1} \left(\int_{B(0,1)} |f|^2 \right)^{1/2} \leq 1, \quad (6.10)$$

then

$$\left(\int_{B(0,\theta)} \left| u_\varepsilon - \int_{B(0,\theta)} u_\varepsilon \right|^2 \right)^{1/2} + \theta \left(\int_{B(0,\theta)} |u_\varepsilon|^2 \right)^{1/2} \leq \frac{\theta^\sigma}{2}. \quad (6.11)$$

To this end we first note that if $u \in H^1(B(0, 1/2); \mathbb{R}^m)$ is a weak solution of

$$-\operatorname{div}(A^0 \nabla u) + \lambda u = 0 \quad \text{in } B(0, 1/2), \quad (6.12)$$

where $\lambda \in [0, 1]$ and A^0 is a constant matrix satisfying the ellipticity condition (2.7), then, for any $\theta \in (0, 1/8)$,

$$\left(\int_{B(0,\theta)} \left| u - \int_{B(0,\theta)} u \right|^2 \right)^{1/2} + \theta \left(\int_{B(0,\theta)} |u|^2 \right)^{1/2} \leq C_0 \theta \left(\int_{B(0,1/2)} |u|^2 \right)^{1/2}, \quad (6.13)$$

where C_0 depends only on d, m and μ . This follows from the interior Lipschitz estimate

$$\|\nabla u\|_{L^\infty(B(0,1/4))} + \|u\|_{L^\infty(B(0,1/4))} \leq C \|u\|_{L^2(B(0,1/2))}$$

for solutions of the elliptic system (6.12) with constant coefficients.

We now choose $\theta \in (0, 1/4)$ so small that $2^{d/2} C_0 \theta < \theta^\sigma/2$. We claim that (6.11) holds for this θ and for some $\varepsilon_0 \in (0, 1/2)$, which depends at most on σ and A .

Suppose this is not the case. Then there exist sequences $\{M^\ell\} \subset \mathcal{A}$, $\{\varepsilon_\ell\} \subset \mathbb{R}_+$, $\{\lambda_\ell\} \subset [0, 1]$, $\{F_\ell\} \subset L^p(B(0, 1); \mathbb{R}^m)$, $\{f_\ell\} \subset L^2(B(0, 1); \mathbb{R}^{m \times d})$, and $\{u_\ell\} \subset H^1(B(0, 1); \mathbb{R}^m)$, such that $\varepsilon_\ell \rightarrow 0$, $0 \leq \lambda_\ell \leq \varepsilon_\ell^2$,

$$-\operatorname{div}(M^\ell(x/\varepsilon_\ell) \nabla u_\ell) + \lambda_\ell u_\ell = F_\ell + \operatorname{div}(f_\ell) \quad \text{in } B(0, 1), \quad (6.14)$$

and

$$\left(\int_{B(0,1)} |u_\ell|^2 \right)^{1/2} + \varepsilon_\ell^{-1} \left(\int_{B(0,1)} |F_\ell|^2 \right)^{1/2} + \varepsilon_\ell^{-1} \left(\int_{B(0,1)} |f_\ell|^2 \right)^{1/2} \leq 1, \quad (6.15)$$

but

$$\left(\int_{B(0,\theta)} \left| u_\ell - \int_{B(0,\theta)} u_\ell \right|^2 \right)^{1/2} + \theta \left(\int_{B(0,\theta)} |u_\ell|^2 \right)^{1/2} > \frac{\theta^\sigma}{2}. \quad (6.16)$$

By passing to subsequences, we may assume that

$$u_\ell \rightharpoonup u \text{ weakly in } L^2(B(0, 1); \mathbb{R}^m). \quad (6.17)$$

By Caccioppoli's inequality the sequence $\{u_\ell\}$ is bounded in $H^1(B(0, 1/2); \mathbb{R}^m)$. By passing to a subsequence, we may further assume that u_ℓ converges to u weakly in $H^1(B(0, 1/2); \mathbb{R}^m)$

and hence strongly in $L^2(B(0, 1/2); \mathbb{R}^m)$. Also note that $\lambda_\ell \rightarrow 0$, and $F_\ell + \operatorname{div} f_\ell$ converges to zero strongly in $H^{-1}(B(0, 1/2); \mathbb{R}^m)$. This allows us to apply Theorem 5.1 to the system (6.14) in $B(0, 1/2)$. It follows that u is a weak solution of $-\operatorname{div}(\hat{A}\nabla u) = 0$ in $B(0, 1/2)$.

Finally, since $u_\ell \rightarrow u$ strongly in $L^2(B(0, 1/2); \mathbb{R}^m)$, by (6.16), we obtain

$$\left(\int_{B(0, \theta)} |u - \int_{B(0, \theta)} u|^2 \right)^{1/2} + \theta \left(\int_{B(0, \theta)} |u|^2 \right)^{1/2} \geq \frac{\theta^\sigma}{2}. \quad (6.18)$$

Similarly, by (6.17) and (6.15),

$$\left(\int_{B(0, 1)} |u|^2 \right)^{1/2} \leq 1. \quad (6.19)$$

On the other hand, it follows from (6.13) and (6.19) that

$$\left(\int_{B(0, \theta)} |u - \int_{B(0, \theta)} u|^2 \right)^{1/2} + \theta \left(\int_{B(0, \theta)} |u|^2 \right)^{1/2} \leq C_0 2^{d/2} \theta. \quad (6.20)$$

This, together with (6.18), gives $C_0 2^{d/2} \theta \geq \theta^\sigma / 2$, which is in contradiction with our choice of θ . The proof is now complete. \square

Lemma 6.6. *Let $\sigma \in (0, 1)$ and ε_0, θ be given by Lemma 6.5. If $0 < \varepsilon < \varepsilon_0 \theta^{k-1}$ for some $k \geq 1$ and u_ε is a weak solution of (6.7) in $B(0, 1)$ for some $M \in \mathcal{A}$ and $\lambda \in [0, \varepsilon_0^2]$, then*

$$\begin{aligned} & \left(\int_{B(0, \theta^k)} |u_\varepsilon - \int_{B(0, \theta^k)} u_\varepsilon|^2 \right)^{1/2} + \theta^k \sqrt{\lambda} \left(\int_{B(0, \theta^k)} |u_\varepsilon|^2 \right)^{1/2} \\ & \leq \theta^{k\sigma} \left\{ \left(\int_{B(0, 1)} |u_\varepsilon - \int_{B(0, 1)} u_\varepsilon|^2 \right)^{1/2} + \sqrt{\lambda} \left(\int_{B(0, 1)} |u_\varepsilon|^2 \right)^{1/2} + I_k + J_k \right\}, \end{aligned}$$

where I_k and J_k are defined by

$$\begin{aligned} I_k &= \varepsilon_0^{-1} \sum_{\ell=0}^{k-1} \theta^{\ell(2-\sigma)} \left(\int_{B(0, \theta^\ell)} |F|^2 \right)^{1/2}, \\ J_k &= \varepsilon_0^{-1} \sum_{\ell=0}^{k-1} \theta^{\ell(1-\sigma)} \left(\int_{B(0, \theta^\ell)} |f|^2 \right)^{1/2}. \end{aligned} \quad (6.21)$$

Proof. The lemma is proved by an induction argument on k . The case $k = 1$ is given by Lemma 6.5. Suppose now that the lemma holds for some $k \geq 1$. Let u_ε be a weak solution of (6.7) in $B(0, 1)$ for some $0 < \varepsilon < \varepsilon_0 \theta^k$. Consider the function $v(x) = u_\varepsilon(\theta^k x)$. Observe that v satisfies the system

$$-\operatorname{div}(M(x/(\theta^{-k}\varepsilon))\nabla v) + \theta^{2k}\lambda v = G + \operatorname{div}(H) \quad \text{in } B(0, 1),$$

where $G(x) = \theta^{2k}F(\theta^k x)$ and $H(x) = \theta^k f(\theta^k x)$. Since $\theta^{-k}\varepsilon < \varepsilon_0$ and $\theta^{2k}\lambda \in [0, \varepsilon_0^2]$, it follows from Lemma 6.5 that

$$\begin{aligned}
& \left(\int_{B(0, \theta^{k+1})} |u_\varepsilon - \int_{B(0, \theta^{k+1})} u_\varepsilon|^2 \right)^{1/2} + \theta^{k+1} \sqrt{\lambda} \left(\int_{B(0, \theta^{k+1})} |u_\varepsilon|^2 \right)^{1/2} \\
&= \left(\int_{B(0, \theta)} |v - \int_{B(0, \theta)} v|^2 \right)^{1/2} + \theta(\sqrt{\theta^{2k}\lambda}) \left(\int_{B(0, \theta)} |v|^2 \right)^{1/2} \\
&\leq \theta^\sigma \left\{ \left(\int_{B(0, 1)} |v - \int_{B(0, 1)} v|^2 \right)^{1/2} + \sqrt{\theta^{2k}\lambda} \left(\int_{B(0, 1)} |v|^2 \right)^{1/2} \right. \\
&\quad \left. + \varepsilon_0^{-1} \theta^{2k} \left(\int_{B(0, 1)} |F(\theta^k x)|^2 dx \right)^{1/2} + \varepsilon_0^{-1} \theta^k \left(\int_{B(0, 1)} |f(\theta^k x)|^2 dx \right)^{1/2} \right\} \\
&\leq \theta^\sigma \left\{ \left(\int_{B(0, \theta^k)} |u_\varepsilon - \int_{B(0, \theta^k)} u_\varepsilon|^2 \right)^{1/2} + \theta^k \sqrt{\lambda} \left(\int_{B(0, \theta^k)} |u_\varepsilon|^2 \right)^{1/2} \right. \\
&\quad \left. + \varepsilon_0^{-1} \theta^{2k} \left(\int_{B(0, \theta^k)} |F|^2 \right)^{1/2} + \varepsilon_0^{-1} \theta^k \left(\int_{B(0, \theta^k)} |f|^2 \right)^{1/2} \right\}.
\end{aligned}$$

By the induction assumption this is bounded by

$$\begin{aligned}
& \theta^{(k+1)\sigma} \left\{ \left(\int_{B(0, 1)} |u_\varepsilon - \int_{B(0, 1)} u_\varepsilon|^2 \right)^{1/2} + \sqrt{\lambda} \left(\int_{B(0, 1)} |u_\varepsilon|^2 \right)^{1/2} \right. \\
&\quad \left. + I_k + J_k + \varepsilon_0^{-1} \theta^{k(2-\sigma)} \left(\int_{B(0, \theta^k)} |F|^2 \right)^{1/2} + \varepsilon_0^{-1} \theta^{k(1-\sigma)} \left(\int_{B(0, \theta^k)} |f|^2 \right)^{1/2} \right\} \\
&= \theta^{(k+1)\sigma} \left\{ \left(\int_{B(0, 1)} |u_\varepsilon - \int_{B(0, 1)} u_\varepsilon|^2 \right)^{1/2} + \sqrt{\lambda} \left(\int_{B(0, 1)} |u_\varepsilon|^2 \right)^{1/2} + I_{k+1} + J_{k+1} \right\},
\end{aligned}$$

where we have used the definitions of I_k and J_k . The proof is complete. \square

We are now ready to give the proof of Theorem 6.1.

Proof of Theorem 6.1. Let ε_0 and θ be given by Lemma 6.5. We may assume that $0 \leq \lambda \leq \varepsilon_0^2 R^{-2}$. The case $\varepsilon_0^2 R^{-2} < \lambda \leq R^{-2}$ follows easily from the case $\lambda = \varepsilon_0^2 R^{-2}$. By translation and dilation we may also assume that $x_0 = 0$ and $R = 1$. Thus $u_\varepsilon \in H^1(B(0, 1); \mathbb{R}^m)$ is a weak solution of $-\operatorname{div}(M(x/\varepsilon)\nabla u_\varepsilon) + \lambda u_\varepsilon = F + \operatorname{div}(f)$ in $B(0, 1)$ for some $M \in \mathcal{A}$, $0 < \varepsilon < 1$ and $\lambda \in [0, \varepsilon_0^2]$. Let $\varepsilon < r < 1$. We may assume that $r < \varepsilon_0 \theta$, as the case $r \geq \varepsilon_0 \theta$ follows directly from Caccioppoli's inequality.

Now we choose $k \geq 1$ so that $\varepsilon_0 \theta^{k+1} \leq r < \varepsilon_0 \theta^k$. It follows from Lemma 6.6 and (6.5)

that

$$\begin{aligned}
& \left(\int_{B(0,r)} |\nabla u_\varepsilon|^2 \right)^{1/2} + \sqrt{\lambda} \left(\int_{B(0,r)} |u_\varepsilon|^2 \right)^{1/2} \\
& \leq C \left\{ \left(\int_{B(0,\theta^k/2)} |\nabla u_\varepsilon|^2 \right)^{1/2} + \sqrt{\lambda} \left(\int_{B(0,\theta^k/2)} |u_\varepsilon|^2 \right)^{1/2} \right\} \\
& \leq C \left\{ \theta^{-k} \left(\int_{B(0,\theta^k)} \left| u_\varepsilon - \int_{B(0,\theta^k)} u_\varepsilon \right|^2 \right)^{1/2} + \sqrt{\lambda} \left(\int_{B(0,\theta^k)} |u_\varepsilon|^2 \right)^{1/2} \right. \\
& \quad \left. + \theta^k \left(\int_{B(0,\theta^k)} |F|^2 \right)^{1/2} + \left(\int_{B(0,\theta^k)} |f|^2 \right)^{1/2} \right\} \\
& \leq C \theta^{k(\sigma-1)} \left\{ \left(\int_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2} + \theta^{k(2-\sigma)} \left(\int_{B(0,\theta^k)} |F|^2 \right)^{1/2} \right. \\
& \quad \left. + I_k + J_k + \theta^{k(1-\sigma)} \left(\int_{B(0,\theta^k)} |f|^2 \right)^{1/2} \right\}.
\end{aligned}$$

Finally, note that by (6.21),

$$\begin{aligned}
I_k & \leq C \sup_{\substack{x \in B(0,1/2) \\ r \leq t \leq 1/2}} t \left(\int_{B(x,t)} |F|^2 \right)^{1/2}, \\
J_k & \leq C \sup_{x \in B(0,1/2)} \left(\int_{B(x,r)} |f|^2 \right)^{1/2}.
\end{aligned}$$

We obtain

$$\begin{aligned}
& \left(\int_{B(0,r)} |\nabla u_\varepsilon|^2 \right)^{1/2} + \sqrt{\lambda} \left(\int_{B(0,r)} |u_\varepsilon|^2 \right)^{1/2} \\
& \leq C_\sigma r^{\sigma-1} \left\{ \left(\int_{B(0,1)} |u_\varepsilon|^2 \right)^{1/2} + \sup_{\substack{x \in B(0,1/2) \\ r \leq t \leq 1/2}} t \left(\int_{B(x,t)} |F|^2 \right)^{1/2} \right. \\
& \quad \left. + \sup_{x \in B(0,1/2)} \left(\int_{B(x,r)} |f|^2 \right)^{1/2} \right\}.
\end{aligned}$$

This finishes the proof (with $1 - \sigma$ in the place of σ). \square

As a corollary of Theorem 6.1, we obtain the following.

Theorem 6.7. *Suppose that $A \in APW^2(\mathbb{R}^d)$ and satisfies the ellipticity condition (1.2). Let $F \in L^2_{\text{loc},\text{unif}}(\mathbb{R}^d; \mathbb{R}^m)$, $f \in L^2_{\text{loc},\text{unif}}(\mathbb{R}^{m \times d})$, and u be the solution of*

$$-\operatorname{div}(A \nabla u) + T^{-2}u = F + \operatorname{div}(f) \quad \text{in } \mathbb{R}^d, \tag{6.22}$$

given by Lemma 3.1. Then there exists $\bar{q} > 2$, depending only on d, m and μ , such that for any $1 \leq r \leq T$ and $\sigma \in (0, 1)$,

$$\|\nabla u\|_{S_r^q} + T^{-1}\|u\|_{S_r^2} \leq C_\sigma \left(\frac{T}{r}\right)^\sigma \left\{ \sup_{r \leq t \leq T} t\|F\|_{S_t^2} + \|f\|_{S_r^q} \right\}, \quad (6.23)$$

where $2 \leq q \leq \bar{q}$ and C_σ depends only on σ and A .

Proof. The case $(T/2) \leq r \leq T$ follows from Lemma 3.1 and does not use the almost periodicity of A . To treat the case $1 \leq r < (T/2)$, we use Theorem 6.1 with $\varepsilon = 1$, $\lambda = T^{-2}$ and $R = T$. This, together with the reverse Hölder estimate (3.4), gives

$$\begin{aligned} & \left(\int_{B(x_0, r)} |\nabla u|^q \right)^{1/q} + \sqrt{\lambda} \left(\int_{B(x_0, r)} |u|^2 \right)^{1/2} \\ & \leq C_\sigma \left(\frac{T}{r}\right)^\sigma \left\{ \frac{1}{T} \left(\int_{B(x_0, T)} |u|^2 \right)^{1/2} + \sup_{\substack{x \in B(x_0, T/2) \\ r \leq t \leq T/2}} t \left(\int_{B(x, t)} |F|^2 \right)^{1/2} \right. \\ & \quad \left. + \sup_{x \in B(x_0, T/2)} \left(\int_{B(x, r)} |f|^q \right)^{1/q} \right\}, \end{aligned}$$

where $2 \leq q \leq \bar{q}$ and $\bar{q} > 2$ depends only on d, m and μ . It follows that

$$\begin{aligned} \|\nabla u\|_{S_r^q} + T^{-1}\|u\|_{S_r^2} & \leq C_\sigma \left(\frac{T}{r}\right)^\sigma \left\{ T^{-1}\|u\|_{S_T^2} + \sup_{r \leq t \leq T} t\|F\|_{S_t^2} + \|f\|_{S_r^q} \right\} \\ & \leq C_\sigma \left(\frac{T}{r}\right)^\sigma \left\{ \sup_{r \leq t \leq T} t\|F\|_{S_t^2} + \|f\|_{S_T^2} + \|f\|_{S_r^q} \right\}, \end{aligned}$$

which leads to (6.23), using $\|f\|_{S_T^2} \leq C\|f\|_{S_r^q}$. \square

Corollary 6.8. *Suppose that $A \in APW^2(\mathbb{R}^d)$ and satisfies the ellipticity condition (1.2). Let $T > 1$ and $\sigma \in (0, 1)$. Then*

$$\begin{aligned} \|\nabla \chi_T\|_{S_r^{\bar{q}}} & \leq C_\sigma \left(\frac{T}{r}\right)^\sigma, \\ \|\nabla(\chi_T - \chi_{2T})\|_{S_r^{\bar{q}}} & \leq C_\sigma \left(\frac{T}{r}\right)^\sigma \sup_{r \leq t \leq T} t\|T^{-2}\chi_{2T}\|_{S_t^2}, \end{aligned} \quad (6.24)$$

for any $1 \leq r \leq T$, where C_σ depends only on σ and A .

Proof. The first inequality in (6.24) follows directly from Theorem 6.7 with $F = 0$ and $f = A\nabla P_j^\beta$. The same argument also gives a rough estimate,

$$T^{-1}\|\chi_T\|_{S_r^2} \leq C_\sigma \left(\frac{T}{r}\right)^\sigma. \quad (6.25)$$

To see the second inequality we let $u = \chi_T - \chi_{2T}$. Then

$$-\operatorname{div}(A\nabla u) + T^{-2}u = -(3/4)T^{-2}\chi_{2T}.$$

By Theorem 6.7 we obtain the second inequality in (6.24). \square

7 A quantitative ergodic argument

In this section we establish some general estimates, which formalize and extend the quantitative ergodic argument in [1], for functions in $APW^2(\mathbb{R}^d)$. These estimates allow us to control the norm $\|g\|_{S_1^2}$ by $\|\nabla g\|_{S_t^2}$ for $t \geq 1$ and the function $\omega_k(g; L, R)$, defined by

$$\omega_k(g; L, R) = \sup_{y_1 \in \mathbb{R}^d} \inf_{|z_1| \leq L} \cdots \sup_{y_k \in \mathbb{R}^d} \inf_{|z_k| \leq L} \|\Delta_{y_1 z_1} \Delta_{y_2 z_2} \cdots \Delta_{y_k z_k}(g)\|_{S_R^2}, \quad (7.1)$$

where $0 < L, R < \infty$ and $k \geq 1$. Throughout this section we will assume that $g, \nabla g \in L_{\text{loc}, \text{unif}}^2(\mathbb{R}^d)$ and

$$\langle g \rangle = \lim_{R \rightarrow \infty} \oint_{B(0, R)} g = 0. \quad (7.2)$$

Let

$$u(x, t) = g * \Phi_t(x) = \int_{\mathbb{R}^d} g(y) \Phi_t(x - y) dy, \quad (7.3)$$

where

$$\Phi_t(y) = t^{d/2} \Phi(y/\sqrt{t}) = c_d t^{-d/2} \exp(-|y|^2/(4\sqrt{t}))$$

is the standard heat kernel.

We begin with a lemma that reduces the estimate of $\|g\|_{S_1^2}$ to that of $\|u(\cdot, 1)\|_\infty$.

Lemma 7.1. *Let $u(x, t) = g * \Phi_t(x)$. Then, for $0 < R < \infty$,*

$$\|g\|_{S_R^2} \leq C \left\{ \|u(\cdot, R^2)\|_\infty + R \|\nabla g\|_{S_R^2} \right\}, \quad (7.4)$$

where C depends only on d .

Proof. Note that if $g(x) = f(Rx)$, then $\|g\|_{S_1^2} = \|f\|_{S_R^2}$ and $\Phi_t * g(x) = \Phi_{tR^2} * f(Rx)$. Thus, by rescaling, we may assume that $R = 1$. We will show that for any $r \geq 1$ and $x \in \mathbb{R}^d$,

$$\left| u(x, 1) - \oint_{B(x, r)} g \right| \leq C r^{\frac{d}{2}+2} \|\nabla g\|_{S_r^2} + C e^{-cr^2} \|g\|_{S_r^2}, \quad (7.5)$$

where $C > 0$ and $c > 0$ depend only on d . Assume (7.5) holds for a moment. Then, by Poincaré inequality, for any $r \geq 1$,

$$\begin{aligned} \|g\|_{S_r^2} &\leq C r \|\nabla g\|_{S_r^2} + \sup_{x \in \mathbb{R}^d} \left| \oint_{B(x, r)} g \right| \\ &\leq C r^{\frac{d}{2}+2} \|\nabla g\|_{S_r^2} + \|u(\cdot, 1)\|_\infty + C e^{-cr^2} \|g\|_{S_r^2}. \end{aligned}$$

We now fix $r > 1$ such that $C e^{-cr^2} \leq (1/2)$. Since $\|g\|_{S_r^2} < \infty$, it follows that

$$\begin{aligned} \|g\|_{S_1^2} &\leq C \|g\|_{S_r^2} \leq C \|u(\cdot, 1)\|_\infty + C \|\nabla g\|_{S_r^2} \\ &\leq C \left\{ \|u(\cdot, 1)\|_\infty + \|\nabla g\|_{S_1^2} \right\}. \end{aligned}$$

It remains to prove (7.5). To this end we first note that

$$\begin{aligned}
\left| u(x, 1) - \int_{B(0, r)} g(x - y) \Phi(y) dy \right| &\leq \int_{\mathbb{R}^d \setminus B(0, r)} |g(x - y)| \Phi(y) dy \\
&\leq \sum_j \int_{Q_j} |g(x - y)| \Phi(y) dy \\
&\leq \sum_j \left(\int_{Q_j} |g(x - \cdot)|^2 \right)^{1/2} \left(\int_{Q_j} |\Phi|^2 \right)^{1/2} \\
&\leq C \|g\|_{S_r^2} \int_{|y| \geq cr} e^{-c|y|^2} dy \\
&\leq C e^{-cr^2} \|g\|_{S_r^2},
\end{aligned}$$

where $\{Q_j\}$ is a collection of non-overlapping cubes with side length cr such that

$$\mathbb{R}^d \setminus B(0, r) \subset \cup_j Q_j \subset \mathbb{R}^d \setminus B(0, r/2).$$

It follows that

$$\begin{aligned}
&\left| u(x, 1) - \int_{B(x, r)} g \right| \\
&\leq C e^{-cr^2} \|g\|_{S_r^2} + \left| \int_{B(x, r)} g - \int_{B(x, r)} g(y) \Phi(x - y) dy \right| \\
&\leq C e^{-cr^2} \|g\|_{S_r^2} + \left| \int_{B(x, r)} \left(g - \int_{B(x, r)} g \right) (\Phi(x - y) - E_r) dy \right| \\
&\quad + \left| \int_{B(x, r)} g \left| \int_{B(0, r)} \Phi - 1 \right| \right|,
\end{aligned} \tag{7.6}$$

where E_r is the average of Φ over $B(0, r)$. By Hölder's and Poincaré inequalities the second term in the r.h.s. of (7.6) is bounded by

$$C r^{d+2} \|\nabla g\|_{S_r^2} \left(\int_{B(0, r)} |\nabla \Phi|^2 \right)^{1/2} \leq C r^{\frac{d}{2}+2} \|\nabla g\|_{S_r^2}.$$

Finally, since $\int_{\mathbb{R}^d} \Phi = 1$, the last term in the r.h.s. of (7.6) is bounded by

$$C \|g\|_{S_r^2} \int_{\mathbb{R}^d \setminus B(0, r)} \Phi \leq C e^{-cr^2} \|g\|_{S_r^2}.$$

This completes the proof of (7.5). \square

To control $\|u(\cdot, t)\|_\infty$, we use a quantitative ergodic result from [1]. We mention that the explicit dependence of constants in k is not used in this paper.

Lemma 7.2. *Let $u(x, t) = g * \Phi_t(x)$, where $g, \nabla g \in L_{\text{loc}, \text{unif}}^2(\mathbb{R}^d)$ and $\langle g \rangle = 0$. Then, for any $t \geq kR^2$ and $0 < L < \infty$,*

$$\begin{aligned}
\|u(\cdot, t)\|_\infty &\leq C^k \left\{ \omega_k(g; L, R) + \exp\left(-\frac{ct}{kL^2}\right) \|g\|_{S_R^2} \right\}, \\
\|\nabla_x u(\cdot, t)\|_\infty &\leq \frac{C^k}{\sqrt{t}} \left\{ \omega_k(g; L, R) + \exp\left(-\frac{ct}{kL^2}\right) \|g\|_{S_R^2} \right\},
\end{aligned} \tag{7.7}$$

where C and c depend only on d .

Proof. By rescaling we may reduce the general case to the case where $R = 1$ and $t \geq k$. In this case the proposition was proved in [1]. We point out that the condition (7.2), together with the assumption that $g \in L^2_{\text{loc}, \text{unif}}(\mathbb{R}^d)$, implies $\int_{B(x, R)} g \rightarrow 0$, as $R \rightarrow \infty$, for any $x \in \mathbb{R}^d$. It follows by the Lebesgue dominated convergence theorem that $\int_{B(0, R)} u(x, t) dx \rightarrow 0$, as $R \rightarrow \infty$, for any $t > 0$. Hence, $\|u(\cdot, t)\|_\infty \leq \sup_{x, y \in \mathbb{R}^d} |u(x, t) - u(y, t)|$. \square

We are ready to state and prove the main result of this section.

Theorem 7.3. *Let $g \in H^1_{\text{loc}}(\mathbb{R}^d)$. Suppose that $g, \nabla g \in L^2_{\text{loc}, \text{unif}}(\mathbb{R}^d)$ and $\langle g \rangle = 0$. Then, for any $T \geq 2$ and $k \geq 1$,*

$$\begin{aligned} \|g\|_{S^2_1} &\leq C \inf_{1 \leq L \leq T} \left\{ \omega_k(g; L, T) + \exp\left(-\frac{cT^2}{L^2}\right) \|g\|_{S^2_T} \right\} \\ &\quad + C \int_1^T \inf_{1 \leq L \leq t} \left\{ \omega_k(\nabla g; L, t) + \exp\left(-\frac{ct^2}{L^2}\right) \|\nabla g\|_{S^2_t} \right\} dt, \end{aligned} \quad (7.8)$$

where $C > 0$ and $c > 0$ depend only on d and k .

Proof. We first note that $\|\nabla g\|_{S^2_1}$ is bounded by the second integral in the r.h.s. of (7.8) over the interval $[1, 2]$. Thus, in view of Lemma 7.1, it suffices to show that $\|u(\cdot, 1)\|_\infty$ is dominated by the r.h.s. of (7.8), where $u(x, t) = g * \Phi_t(x)$. To this end we use the heat equation $\partial_t u = \Delta_x u$ to obtain

$$\begin{aligned} \|u(\cdot, 1)\|_\infty &\leq \|u(\cdot, T^2)\|_\infty + \int_1^{T^2} \|\partial_s u(\cdot, s)\|_\infty ds \\ &\leq \|u(\cdot, T^2)\|_\infty + \int_1^{T^2} \|\nabla_x^2 u(\cdot, s)\|_\infty ds. \end{aligned} \quad (7.9)$$

By the first inequality in (7.7) with $R = cT$,

$$\|u(\cdot, T^2)\|_\infty \leq C \inf_{1 \leq L \leq T} \left\{ \omega_k(g; L, T) + \exp\left(-\frac{cT^2}{L^2}\right) \|g\|_{S^2_T} \right\}. \quad (7.10)$$

To handle $\|\nabla_x^2 u(\cdot, s)\|_\infty$, we use

$$\nabla_x^2 u = \nabla_x (\nabla g * \Phi_t)$$

and the second inequality in (7.7) with $R = c\sqrt{s}$ to obtain

$$\|\nabla^2 u(\cdot, s)\|_\infty \leq C \inf_{1 \leq L \leq \sqrt{s}} \left\{ \omega_k(\nabla g; L, \sqrt{s}) + \exp\left(-\frac{cs}{L^2}\right) \|\nabla g\|_{S^2_{\sqrt{s}}} \right\} \frac{1}{\sqrt{s}}. \quad (7.11)$$

The estimate (7.8) follows by combining (7.9), (7.10) and (7.11) and using a change of variable $t = \sqrt{s}$ in the integral. \square

Remark 7.4. Suppose that $g \in APW^2(\mathbb{R}^d)$ and $\langle g \rangle = 0$. Then $\omega_k(g, L, L) \rightarrow 0$ as $L \rightarrow \infty$. It follows that the first term in (7.8) goes to zero as $T \rightarrow \infty$. This gives

$$\|g\|_{S^2_1} \leq C \int_1^\infty \inf_{1 \leq L \leq t} \left\{ \omega_k(\nabla g; L, t) + \exp\left(-\frac{ct^2}{L^2}\right) \|\nabla g\|_{S^2_t} \right\} dt, \quad (7.12)$$

where $C > 0$ and $c > 0$ depend only on d and k .

8 Estimates of approximate correctors, part II

In this section we give the proof of Theorems 1.1 and 1.2. Let

$$P = P_k = \{(y_1, z_1), (y_2, z_2), \dots, (y_k, z_k)\},$$

where $(y_i, z_i) \in \mathbb{R}^d \times \mathbb{R}^d$. Recall that

$$\Delta_P(f) = \Delta_{y_1 z_1} \Delta_{y_2 z_2} \cdots \Delta_{y_k z_k}(f) \quad (8.1)$$

(if $k = 0$, then $P = \emptyset$ and $\Delta_P(f) = f$). Using the observation that

$$\Delta_{yz}(fg)(x) = \Delta_{yz}(f)(x) \cdot g(x + y) + f(x + z) \cdot \Delta_{yz}(g)(x), \quad (8.2)$$

an induction argument yields

$$\Delta_P(fg)(x) = \sum_{Q \subset P} \Delta_Q(f)(x + z_{j_1} + \cdots + z_{j_\ell}) \cdot \Delta_{P \setminus Q}(g)(x + y_{i_1} + \cdots + y_{j_{i_\ell}}), \quad (8.3)$$

where the sum is taken over all 2^k subsets $Q = \{(y_{i_1}, z_{i_1}), \dots, (y_{i_\ell}, z_{i_\ell})\}$ of P , with $P \setminus Q = \{(y_{j_1}, z_{j_1}), \dots, (y_{j_t}, z_{j_t})\}$. Here, $i_1 < i_2 < \cdots < i_\ell$, $j_1 < j_2 < \cdots < j_t$, and $\ell + t = k$. It follows from (8.3) by Hölder's inequality that

$$\|\Delta_P(fg)\|_{S_R^{q_1}} \leq \sum_{Q \subset P} \|\Delta_Q(f)\|_{S_R^p} \|\Delta_{P \setminus Q}(g)\|_{S_R^q}, \quad (8.4)$$

where $\frac{1}{q_1} \geq \frac{1}{p} + \frac{1}{q}$.

Lemma 8.1. *Suppose that $A \in APW^2(\mathbb{R}^d)$ and satisfies the condition (1.2). Let $F \in L_{\text{loc,unif}}^2(\mathbb{R}^d; \mathbb{R}^m)$, $f \in L_{\text{loc,unif}}^2(\mathbb{R}^{m \times d})$, and u be the solution of (6.22), given by Lemma 3.1. Let $k \geq 0$ and $P = P_k$. Then there exists $\bar{q} > 2$, depending only on d, m and μ , such that any $1 \leq r \leq T$ and $\sigma \in (0, 1)$,*

$$\begin{aligned} & \|\Delta_P(\nabla u)\|_{S_r^q} + T^{-1} \|\Delta_P(u)\|_{S_r^2} \\ & \leq C_\sigma \left(\frac{T}{r}\right)^\sigma \left\{ \sup_{r \leq t \leq T} t \|\Delta_P(F)\|_{S_t^2} + \|\Delta_P(f)\|_{S_r^{q_0}} \right\} \\ & \quad + C_\sigma \left(\frac{T}{r}\right)^\sigma \sum_{P=Q_0 \cup Q_1 \cup \cdots \cup Q_\ell} \|\Delta_{Q_1} A\|_{S_r^p} \cdots \|\Delta_{Q_\ell} A\|_{S_r^p} \\ & \quad \cdot \left\{ \sup_{r \leq t \leq T} t \|\Delta_{Q_0}(F)\|_{S_t^2} + \|\Delta_{Q_0}(f)\|_{S_r^{q_0}} \right\}, \end{aligned} \quad (8.5)$$

where $2 \leq q < q_0 \leq \bar{q}$, $\frac{1}{q} - \frac{1}{q_0} \geq \frac{k}{p}$, and C_σ depends only on d, m, k, σ and A . The sum in (8.5) is taken over all partitions of $P = Q_0 \cup Q_1 \cup \cdots \cup Q_\ell$ with $1 \leq \ell \leq k - 1$ and $Q_j \neq \emptyset$.

Proof. Let $\bar{q} > 2$ be the same as in Theorem 6.7. We prove the estimate (8.5) by an induction argument on k . Note that the case $k = 0$ with $P = \emptyset$ is given by Corollary 6.7. Let $k \geq 1$ and suppose the estimate (8.5) holds for $P = P_\ell$ with $0 \leq \ell \leq k - 1$. Let

$2 \leq q < q_0 \leq \bar{q}$ and $\frac{1}{q} - \frac{1}{q_0} \geq \frac{k}{p}$. By applying Δ_P to the system (6.22) and using (8.3), we obtain

$$\begin{aligned} & -\operatorname{div}(A(\cdot + z_1 + \cdots + z_k) \nabla \Delta_P(u)) + T^{-2} \Delta_P(u) \\ & = \Delta_P(F) + \operatorname{div}(\Delta_P(f)) + \operatorname{div}\left(\sum_{Q \subset P, Q \neq \emptyset} \Delta_Q(A) \cdot \Delta_{P \setminus Q}(\nabla u)\right). \end{aligned}$$

It follows from Theorem 6.7 and Hölder's inequality that

$$\begin{aligned} & \|\Delta_P(\nabla u)\|_{S_r^q} + T^{-1} \|\Delta_P(u)\|_{S_r^2} \\ & \leq C_\sigma \left(\frac{T}{r}\right)^{\frac{\sigma}{2}} \left\{ \sup_{r \leq t \leq T} t \|\Delta_P(F)\|_{S_t^2} + \|\Delta_P(f)\|_{S_r^q} \right. \\ & \quad \left. + \sum_{Q \subset P, Q \neq \emptyset} \|\Delta_Q(A)\|_{S_r^p} \|\Delta_{P \setminus Q}(\nabla u)\|_{S_r^{q_1}} \right\}, \end{aligned} \quad (8.6)$$

where q_1 is chosen so that $2 \leq q < q_1 < q_0 \leq \bar{q}$, $\frac{1}{q} - \frac{1}{q_1} \geq \frac{1}{p}$ and $\frac{1}{q_1} - \frac{1}{q_0} \geq \frac{k-1}{p}$. By the induction assumption,

$$\begin{aligned} & \|\Delta_{P \setminus Q}(\nabla u)\|_{S_r^{q_1}} + T^{-1} \|\Delta_{P \setminus Q}(u)\|_{S_r^2} \\ & \leq C_\sigma \left(\frac{T}{r}\right)^{\frac{\sigma}{2}} \left\{ \sup_{r \leq t \leq T} t \|\Delta_{P \setminus Q}(F)\|_{S_t^2} + \|\Delta_{P \setminus Q}(f)\|_{S_r^{q_0}} \right\} \\ & \quad + C_\sigma \left(\frac{T}{r}\right)^{\frac{\sigma}{2}} \sum_{P \setminus Q = Q_0 \cup Q_1 \cup \cdots \cup Q_\ell} \|\Delta_{Q_1} A\|_{S_r^p} \cdots \|\Delta_{Q_\ell} A\|_{S_r^p} \\ & \quad \cdot \left\{ \sup_{r \leq t \leq T} t \|\Delta_{Q_0}(F)\|_{S_t^2} + \|\Delta_{Q_0}(f)\|_{S_r^{q_0}} \right\}. \end{aligned} \quad (8.7)$$

The desired estimate now follows by combining (8.6) and (8.7). \square

Remark 8.2. If $r \geq T$, the argument in the proof of Lemma 8.1, together with the estimate in Lemma 3.2, gives

$$\begin{aligned} & \|\Delta_P(\nabla u)\|_{S_r^q} + T^{-1} \|\Delta_P(u)\|_{S_r^2} \\ & \leq C \left\{ T \|\Delta_P(F)\|_{S_r^2} + \|\Delta_P(f)\|_{S_r^{q_0}} \right\} \\ & \quad + C \sum_{P = Q_0 \cup Q_1 \cup \cdots \cup Q_\ell} \|\Delta_{Q_1} A\|_{S_r^p} \cdots \|\Delta_{Q_\ell} A\|_{S_r^p} \\ & \quad \cdot \left\{ T \|\Delta_{Q_0}(F)\|_{S_r^2} + \|\Delta_{Q_0}(f)\|_{S_r^{q_0}} \right\}, \end{aligned} \quad (8.8)$$

where $2 \leq q < q_0 \leq \bar{q}$, $\frac{1}{q} - \frac{1}{q_0} \geq \frac{k}{p}$, and C depends only on d , m and μ .

Let $\rho_k(L, R)$ be the function defined by (2.18).

Lemma 8.3. Suppose that $A \in APW^2(\mathbb{R}^d)$ and satisfies the condition (1.2). Let $T \geq 1$. Then, for any $\sigma \in (0, 1)$ and $k \geq 1$,

$$\omega_k(\nabla \chi_T; L, R) + \omega_k(T^{-1} \chi_T; L, R) \leq C_\sigma \left(\frac{T}{R}\right)^\sigma \rho_k(L, R), \quad (8.9)$$

where $1 \leq R \leq T$, $0 < L < \infty$, and C_σ depends only on σ , k , and A . If $R \geq T$, we have

$$\omega_k(\nabla \chi_T; L, R) + \omega_k(T^{-1} \chi_T; L, R) \leq C \rho_k(L, R), \quad (8.10)$$

where C depends only on d , m and μ .

Proof. In view of the definition of χ_T , estimates (8.9) and (8.10) follow directly from Lemma 8.1 and Remark 8.2, respectively, with $q = 2$, $q_0 = \bar{q}$, and $\frac{k}{p} = \frac{1}{2} - \frac{1}{\bar{q}}$. \square

We are now in a position to give the proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. We use Theorem 7.3 with $g = \chi_T$ and $t = T^2$. Note that by (3.6) and (6.24),

$$\|g\|_{S_T^2} \leq C T,$$

and for any $\sigma \in (0, 1)$,

$$\|\nabla g\|_{S_T^2} \leq C_\sigma \left(\frac{T}{r}\right)^\sigma,$$

for $1 \leq r \leq T$. In particular, we have $\|\nabla \chi_T\|_{S_1^2} \leq C_\sigma T^\sigma$. Also, by Lemma 8.3,

$$T^{-1} \omega_k(g; L, T) \leq C \rho_k(L, T),$$

and for any $\sigma \in (0, 1)$ and $1 \leq t \leq T$,

$$\omega_k(\nabla g; L, t) \leq C_\sigma \left(\frac{T}{t}\right)^\sigma \rho_k(L, t).$$

It follows by Theorem 7.3 that

$$\begin{aligned} \|\chi_T\|_{S_1^2} &\leq C T \inf_{1 \leq L \leq T} \left\{ \rho_k(L, T) + \exp\left(-\frac{cT^2}{L^2}\right) \right\} \\ &\quad + C \int_1^T \inf_{1 \leq L \leq t} \left\{ \rho_k(L, t) + \exp\left(-\frac{ct^2}{L^2}\right) \right\} \left(\frac{T}{t}\right)^\sigma dt. \end{aligned} \quad (8.11)$$

It is not hard to see that the first term in the r.h.s. of (8.11) is bounded by the integral in (8.11) from $(T/2)$ to T . As a result, the estimate (1.8) follows. \square

Remark 8.4. Suppose that there exist some $k \geq 1$, $0 < \alpha \leq 1$ and $C > 0$ such that

$$\rho_k(L, L) \leq C L^{-\alpha} \quad \text{for any } L \geq 1. \quad (8.12)$$

By choosing $L = t^\delta$ for some $\delta \in (0, 1)$, we see that

$$\inf_{1 \leq L \leq t} \left\{ \rho_k(L, L) + \exp\left(-\frac{ct^2}{L^2}\right) \right\} \leq \frac{C}{t^{\delta\alpha}}.$$

It follows from (1.8) that $\|\chi_T\|_{S_1^2} \leq C T^{1-\delta\alpha}$. Since $\delta \in (0, 1)$ is arbitrary, we obtain

$$\|\chi_T\|_{S_1^2} \leq C_\beta T^{1-\beta}, \quad (8.13)$$

for any $\beta \in (0, \alpha)$ and $T \geq 1$.

Proof of Theorem 1.2. Suppose that there exist $k \geq 1$, $\delta > 0$ and $C > 0$ such that

$$\rho_k(L, L) \leq CL^{-1-\delta} \quad \text{for any } L \geq 1. \quad (8.14)$$

It follows by Remark 8.4 that $\|\chi_T\|_{S_1^2} \leq C_\sigma T^\sigma$ for any $T \geq 1$ and $\sigma \in (0, 1)$. Let $g = \chi_T - \chi_{2T}$. Note that by Corollary 6.8,

$$\|\nabla g\|_{S_1^2} \leq C_\sigma T^{\sigma-1} \quad \text{for any } T \geq 1 \text{ and } \sigma \in (0, 1). \quad (8.15)$$

We will show that there exists some $\beta > 0$ such that

$$\|g\|_{S_1^2} \leq CT^{-\beta} \quad \text{for any } T \geq 1. \quad (8.16)$$

This would imply that $\{\chi_{2^j}, j = 1, 2, \dots\}$ is a Cauchy sequence in the Banach space $S_1^2 = \{F \in L_{\text{loc}}^2(\mathbb{R}^d) : \|F\|_{S_1^2} < \infty\}$. Let χ be the limit of χ_{2^j} in S_1^2 . It is easy to see that $\|\chi\|_{S_1^2} + \|\nabla \chi\|_{S_1^2} \leq C$. Since $\chi_T \in APW^2(\mathbb{R}^d)$ and $\|g\|_{W^2} \leq \|g\|_{S_1^2}$, we also obtain $\chi \in APW^2(\mathbb{R}^d)$. Note that (8.16) also gives $\|\chi_T\|_{S_1^2} \leq C$.

To see (8.16), we let $u(x, t) = g * \Phi_t(x)$. In view of Lemma 7.1 and (8.15), it suffices to show that

$$\|u(\cdot, 1)\|_\infty \leq CT^{-\beta} \quad (8.17)$$

for some $\beta > 0$. To this end we note that since $g \in APW^2(\mathbb{R}^d)$, $\omega_1(g; L, L) \rightarrow 0$ as $L \rightarrow \infty$. It follows by Lemma 7.2 that $\|u(\cdot, t)\|_\infty \rightarrow 0$ as $t \rightarrow \infty$. Thus, as in the proof of Theorem 7.3,

$$\begin{aligned} \|u(\cdot, 1)\|_\infty &\leq \int_1^\infty \|\partial_t u(\cdot, t)\|_\infty dt \\ &\leq \int_1^\infty \|\nabla_x(\nabla g * \Phi_t)\|_\infty dt \\ &\leq Ct_0 \|\nabla g\|_{S_1^2} + \int_{t_0^2}^\infty \|\nabla_x(\nabla g * \Phi_t)\|_\infty dt \\ &\leq Ct_0 T^{\sigma-1} + \int_{t_0^2}^\infty \left\{ \|\nabla_x(\nabla \chi_T * \Phi_t)\|_\infty + \|\nabla_x(\nabla \chi_{2T} * \Phi_t)\|_\infty \right\} dt, \end{aligned} \quad (8.18)$$

where $t_0 > 1$ is to be chosen and we have used the estimate

$$\|\nabla_x(\nabla g * \Phi_t)\|_\infty \leq Ct^{-1/2} \|\nabla g\|_{S_1^2}$$

for the third inequality and (8.15) for the fourth.

As in the proof of Theorem 1.1, the integral in the r.h.s. of (8.18) is bounded by

$$\begin{aligned} &CT^\sigma \int_{t_0^2}^\infty \inf_{1 \leq L \leq \sqrt{t}} \left\{ \rho_k(L, \sqrt{t}) + \exp\left(-\frac{ct}{L^2}\right) \right\} \frac{dt}{\sqrt{t}} \\ &\leq CT^\sigma \int_{t_0}^\infty \inf_{1 \leq L \leq t} \left\{ \rho_k(L, t) + \exp\left(-\frac{ct^2}{L^2}\right) \right\} dt \\ &\leq CT^\sigma \int_{t_0}^\infty \inf_{1 \leq L \leq t} \left\{ \frac{1}{L^{1+\delta}} + \exp\left(-\frac{ct^2}{L^2}\right) \right\} dt, \end{aligned}$$

where we have used the condition (8.14) for the last step. By choosing $L = t^\alpha$ for $\alpha \in (0, 1)$, it follows that the integral in the r.h.s. of (8.18) is bounded by $CT^\sigma t_0^{1-(1+\delta)\alpha}$. As a result, we have proved that

$$\|u(\cdot, 1)\|_\infty \leq C t_0 T^{\sigma-1} + C T^\sigma t_0^{1-(1+\delta)\alpha} = C t_0 T^\sigma \left\{ T^{-1} + t_0^{-(1+\delta)\alpha} \right\}.$$

Finally, we choose $t_0 > 1$ such that $t_0^{(1+\delta)\alpha} = T$. This gives

$$\|u(\cdot, 1)\|_\infty \leq C t_0 T^{\sigma-1} = C T^{\sigma-1+\frac{1}{(1+\delta)\alpha}} = C T^{-\beta},$$

where

$$\beta = 1 - \sigma - \frac{1}{(1+\delta)\alpha} > 0,$$

if $\sigma > 0$ is small and α is close to 1. This completes the proof. \square

9 Estimates of dual approximate correctors

Let $\chi_T = (\chi_{T,j}^{\alpha\beta})$ be the approximate correctors defined by (3.5). For $1 \leq i, j \leq d$ and $1 \leq \alpha, \beta \leq m$, let $b_T = A + A \nabla \chi_T - \hat{A} = (b_{T,ij}^{\alpha\beta})$ with

$$b_{T,ij}^{\alpha\beta}(y) = a_{ij}^{\alpha\beta}(y) + a_{ik}^{\alpha\gamma}(y) \frac{\partial}{\partial y_k} (\chi_{T,j}^{\gamma\beta}(y)) - \hat{a}_{ij}^{\alpha\beta}. \quad (9.1)$$

To establish the convergence rates in Theorem 1.4, as in [17], we introduce the matrix-valued function $\phi_T = (\phi_{T,ij}^{\alpha\beta})$, called the dual approximate correctors and defined by the following auxiliary equations:

$$-\Delta \phi_{T,ij}^{\alpha\beta} + T^{-2} \phi_{T,ij}^{\alpha\beta} = b_{T,ij}^{\alpha\beta} - \langle b_{T,ij}^{\alpha\beta} \rangle, \quad (9.2)$$

where $\phi_{T,ij}^{\alpha\beta} \in H_{\text{loc}}^1(\mathbb{R}^d)$ are the weak solutions given by Lemma 3.1. In this section we establish the uniform local L^2 estimates for ϕ_T and its derivatives.

Throughout the section we assume that $A \in APW^2(\mathbb{R}^d)$ and satisfies the ellipticity condition (1.2). It follows that $\nabla \chi_T \in APW^2(\mathbb{R}^d)$ and thus $b_T \in APW^2(\mathbb{R}^d)$. Moreover, by (6.24), for any $\sigma \in (0, 1)$ and $1 \leq R \leq T$,

$$\|b_T\|_{S_R^2} \leq C_\sigma \left(\frac{T}{R} \right)^\sigma, \quad (9.3)$$

where C_σ depends only on σ and A .

Lemma 9.1. *Let $k \geq 1$ and $\sigma \in (0, 1)$. Then, for $0 < L < \infty$ and $1 \leq R \leq T$,*

$$\omega_k(b_T; L, R) \leq C_\sigma \left(\frac{T}{R} \right)^\sigma \rho_k(L, R), \quad (9.4)$$

where C_σ depends only on σ , k and A .

Proof. Let $\frac{k}{p} = \frac{1}{2} - \frac{1}{\bar{q}}$, where $\bar{q} > 2$ is given by (3.4). Note that

$$\Delta_P(b_T) = \Delta_P(A) + \sum_{Q \subset P} \Delta_Q(A) \cdot \Delta_{P \setminus Q}(\nabla \chi_T).$$

It follows by Hölder's inequality that if $\frac{k-1}{p} + \frac{1}{q} = \frac{1}{2}$,

$$\begin{aligned} \|\Delta_P(b_T)\|_{S_R^2} &\leq \|\Delta_P(A)\|_{S_R^2} + \|A\|_\infty \|\Delta_P(\nabla \chi_T)\|_{S_R^2} \\ &\quad + \sum_{Q \subset P, Q \neq \emptyset} \|\Delta_Q(A)\|_{S_R^p} \|\Delta_{P \setminus Q}(\nabla \chi_T)\|_{S_R^q} \\ &\leq C_\sigma \left(\frac{T}{R}\right)^\sigma \sum_{Q_1 \cup Q_2 \cup \dots \cup Q_\ell = P} \|\Delta_{Q_1} A\|_{S_R^p} \|\Delta_{Q_2}(A)\|_{S_R^p} \cdots \|\Delta_{Q_\ell}(A)\|_{S_R^p}, \end{aligned}$$

where we have used Lemma 8.1 with $q_0 = 2$ for the last step. By applying

$$\sup_{y_1 \in \mathbb{R}^d} \inf_{|z_1| \leq L} \cdots \sup_{y_k \in \mathbb{R}^d} \inf_{|z_k| \leq L}$$

to the both sides of the inequality above, we obtain (9.4). \square

Lemma 9.2. Assume $F \in L_{\text{loc}}^2, \text{unif}(\mathbb{R}^d)$. Let $u \in H_{\text{loc}}^2(\mathbb{R}^d)$ be the weak solution of

$$-\Delta u + T^{-2}u = F \quad \text{in } \mathbb{R}^d, \quad (9.5)$$

given by Lemma 3.1. Then for any $0 < R < \infty$,

$$T^{-1} \|\nabla u\|_{S_R^2} + T^{-2} \|u\|_{S_R^2} \leq C \|F\|_{S_R^2}, \quad (9.6)$$

where C depends only on d . Furthermore,

$$\|\nabla^2 u\|_{S_R^2} \leq C \log \left(2 + \frac{T}{R}\right) \|F\|_{S_R^2}. \quad (9.7)$$

Proof. By rescaling we may assume that $T = 1$. We may also assume that $d \geq 3$. The case $d = 2$ may be handled by the method of descending (introducing a dummy variable and considering the equation in \mathbb{R}^3).

To show (9.6), we write

$$u(x) = \int_{\mathbb{R}^d} \Gamma(y) F(x - y) dy,$$

where $\Gamma(x)$ denotes the fundamental solution for the operator $-\Delta + 1$ in \mathbb{R}^d , with pole at the origin. Using Minkowski's inequality, we see that

$$\|u\|_{S_R^2} \leq \int_{\mathbb{R}^d} |\Gamma(y)| \|F\|_{S_R^2} dy \leq C \|F\|_{S_R^2},$$

where the last inequality follows from the estimate $|\Gamma(x)| \leq C |x|^{2-d} e^{-c|x|}$. Similarly, by using the estimate $|\nabla \Gamma(x)| \leq C |x|^{1-d} e^{-c|x|}$, we obtain

$$\|\nabla u\|_{S_R^2} \leq \int_{\mathbb{R}^d} |\nabla \Gamma(y)| \|F\|_{S_R^2} dy \leq C \|F\|_{S_R^2}.$$

Finally, to see (9.7), we fix $B = B(x_0, R)$ and choose $\varphi \in C_0^1(3B)$ such that $\varphi = 1$ in $2B$. Write

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^d} \Gamma(x - y) \varphi(y) F(y) dy + \int_{\mathbb{R}^d} \Gamma(x - y) (1 - \varphi(y)) F(y) dy \\ &= u_1(x) + u_2(x). \end{aligned}$$

By the well known singular integral estimates,

$$\left(\int_B |\nabla^2 u_1|^2 \right)^{1/2} \leq C \left(\int_{3B} |F|^2 \right)^{1/2} \leq C \|F\|_{S_R^2}. \quad (9.8)$$

Using the estimate $|\nabla^2 \Gamma(x)| \leq C|x|^{-d}e^{-c|x|}$, we obtain that, for any $x \in B$,

$$\begin{aligned} |\nabla^2 u_2(x)| &\leq C \int_{(2B)^c} |y - x_0|^{-d} e^{-c|y-x_0|} |F(y)| dy \\ &\leq C \sum_{j=1}^{\infty} e^{-c2^j R} \int_{|y-x_0| \leq 2^j R} |F(y)| dy \\ &\leq C \|F\|_{S_R^2} \sum_{j=1}^{\infty} e^{-c2^j R} \\ &\leq C \|F\|_{S_R^2} \log(2 + R^{-1}). \end{aligned}$$

This, together with (9.8), gives

$$\left(\int_B |\nabla^2 u|^2 \right)^{1/2} \leq C \|F\|_{S_R^2} \log(2 + R^{-1})$$

for any $B = B(x_0, R)$. The estimate (9.7) now follows. \square

Remark 9.3. It follows from Lemma 9.2 and estimate (9.3) that for $1 \leq R \leq T$ and $\sigma \in (0, 1)$,

$$T^{-2} \|\phi_T\|_{S_R^2} + T^{-1} \|\nabla \phi_T\|_{S_R^2} + \|\nabla^2 \phi_T\|_{S_R^2} \leq C_\sigma \left(\frac{T}{R} \right)^\sigma, \quad (9.9)$$

where C_σ depends only on σ and A .

Lemma 9.4. *Let $k \geq 1$ and $\sigma \in (0, 1)$. Then, for $1 \leq R \leq T$ and $0 < L < \infty$,*

$$\begin{aligned} T^{-2} \omega_k(\phi_T; L, R) + T^{-1} \omega_k(\nabla \phi_T; L, R) + \omega_k(\nabla^2 \phi_T; L, R) \\ \leq C_\sigma \left(\frac{T}{R} \right)^\sigma \rho_k(L, R), \end{aligned} \quad (9.10)$$

where C_σ depends only on σ , k and A .

Proof. Let $u = \phi_T$. Since the difference operator Δ_P commutes with Δ , in view of Lemma 9.2, we have

$$T^{-2} \|\Delta_P(u)\|_{S_R^2} + T^{-1} \|\Delta_P(\nabla u)\|_{S_R^2} \leq C \|\Delta_P(b_T)\|_{S_R^2}.$$

It follows that

$$T^{-2} \omega_k(u; L, R) + T^{-1} \omega_k(\nabla u; L, R) \leq C \omega_k(b_T, L, R).$$

Similarly,

$$\omega_k(\nabla^2 u; L, R) \leq C \log(2 + TR^{-1}) \omega_k(b_T; L, R).$$

The desired estimates now follows from (9.4). \square

We are now ready to state and prove our main estimates for the dual approximate correctors.

Theorem 9.5. Let $\phi_T = (\phi_{T,ij}^{\alpha\beta})$ be defined in (9.2). Let $k \geq 1$ and $\sigma \in (0, 1)$. Then there exists $c > 0$, depending only on d and k , such that for any $T \geq 2$ and $\sigma \in (0, 1)$,

$$\begin{aligned} & T^{-1} \|\phi_T\|_{S_1^2} + \|\nabla \phi_T\|_{S_1^2} \\ & \leq C_\sigma \int_1^T \inf_{1 \leq L \leq t} \left\{ \rho_k(L, t) + \exp\left(\frac{-ct^2}{L^2}\right) \right\} \left(\frac{T}{t}\right)^\sigma dt, \end{aligned} \quad (9.11)$$

where C_σ depends only on k , σ and A .

Proof. With estimates (9.9) and (9.10) at our disposal, as in the case of (1.8), this theorem follows readily from Theorem 7.3. \square

Let

$$h_{T,j}^{\alpha\beta} = \frac{\partial}{\partial x_i} \left(\phi_{T,ij}^{\alpha\beta} \right), \quad (9.12)$$

where $\phi_T = (\phi_{T,ij}^{\alpha\beta})$ is defined in (9.2). Note that the index i is summed.

Theorem 9.6. Let $h_T = (h_{T,j}^{\alpha\beta})$ be defined by (9.12). Then

$$T \|\nabla h_T\|_{S_1^2} \leq C \|\chi_T\|_{S_1^2}, \quad (9.13)$$

where C depends only on d .

Proof. Observe that by the definition of χ_T ,

$$\frac{\partial}{\partial x_i} (b_{T,ij}^{\alpha\beta}) = T^{-2} \chi_{T,j}^{\alpha\beta},$$

for each $1 \leq j \leq d$ and $1 \leq \alpha, \beta \leq m$ (index i is summed). In view of (9.2) this gives

$$-\Delta h_T + T^{-2} h_T = T^{-2} \chi_T \quad \text{in } \mathbb{R}^d.$$

As a result, estimate (9.13) follows readily from Lemma 9.2. \square

10 Convergence rates

In this section we give the proof of Theorem 1.2, which establishes the near optimal convergence rate in L^2 . Our approach follows the same line of argument as in [16, 18], which in turn use ideas from [19]. While the papers [19, 16, 18] all deal with the case of periodic coefficients, our argument relies on the estimates for approximate correctors in Theorem 1.1 as well as estimates for dual approximate correctors in Section 8.

We begin by introducing smoothing operators S_ε and $K_{\varepsilon,\delta}$. Let $\zeta \in C_0^\infty(B(0, 1))$ be a nonnegative function with $\int_{\mathbb{R}^d} \zeta = 1$, and $\zeta_\varepsilon(x) = \varepsilon^{-d} \zeta(x/\varepsilon)$. Define

$$S_\varepsilon f(x) = \zeta_\varepsilon * f(x) = \int_{\mathbb{R}^d} \zeta_\varepsilon(y) f(x - y) dy. \quad (10.1)$$

Note that, for $1 \leq p \leq \infty$,

$$\|S_\varepsilon f\|_{L^p(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}. \quad (10.2)$$

It is known that if $f \in L^p(\mathbb{R}^d)$ and $g \in L^p_{\text{loc}, \text{unif}}(\mathbb{R}^d)$, then

$$\|g(x/\varepsilon)S_\varepsilon(f)\|_{L^p(\mathbb{R}^d)} \leq \sup_{x \in \mathbb{R}^d} \left(\int_{B(x,1)} |g|^p \right)^{1/p} \|f\|_{L^p(\mathbb{R}^d)}, \quad (10.3)$$

and for $f \in W^{1,p}(\mathbb{R}^d)$,

$$\|S_\varepsilon(f) - f\|_{L^p(\mathbb{R}^d)} \leq C\varepsilon \|\nabla f\|_{L^p(\mathbb{R}^d)}, \quad (10.4)$$

where C depends only on d (see e.g. [16] for a proof of (10.3)-(10.4)).

Let $\delta \geq 2\varepsilon$ be a small parameter to be determined. Let $\eta_\delta \in C_0^\infty(\Omega)$ be a cut-off function so that $\eta_\delta(x) = 0$ in $\Omega_\delta = \{x \in \Omega; \text{dist}(x, \partial\Omega) < \delta\}$, $\eta_\delta(x) = 1$ in $\Omega \setminus \Omega_{2\delta}$ and $|\nabla \eta_\delta| \leq C\delta^{-1}$. Define

$$K_{\varepsilon, \delta} f(x) = S_\varepsilon(\eta_\delta f)(x). \quad (10.5)$$

Note that $K_{\varepsilon, \delta} f \in C_0^\infty(\Omega)$, as $\delta \geq 2\varepsilon$.

Lemma 10.1. *Let Ω be a bounded Lipschitz domain. Then, for any $u \in H^1(\mathbb{R}^d)$,*

$$\int_{\Omega_\varepsilon} |u|^2 \leq C\varepsilon \|u\|_{H^1(\mathbb{R}^d)} \|u\|_{L^2(\mathbb{R}^d)}, \quad (10.6)$$

where $\Omega_\varepsilon = \{x \in \Omega : \text{dist}(x, \partial\Omega) < \varepsilon\}$ and the constant C depends only on Ω .

Proof. See e.g. [18]. □

Lemma 10.2. *Let $u_\varepsilon, u_0 \in H^1(\Omega; \mathbb{R}^m)$ be weak solutions of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ and $\mathcal{L}_0(u_0) = F$, respectively, in Ω . Assume further that $u_0 \in H^2(\Omega; \mathbb{R}^m)$. Set*

$$w_\varepsilon = u_\varepsilon - u_0 - \varepsilon \chi_{T,k}^\beta(x/\varepsilon) K_{\varepsilon, \delta} \left(\frac{\partial u_0^\beta}{\partial x_k} \right), \quad (10.7)$$

where $T = \varepsilon^{-1}$. Then

$$\begin{aligned} \mathcal{L}_\varepsilon(w_\varepsilon) = & \frac{\partial}{\partial x_i} \left\{ \left\{ \hat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon) \right\} \left\{ K_{\varepsilon, \delta} \left(\frac{\partial u_0^\beta}{\partial x_j} \right) - \frac{\partial u_0^\beta}{\partial x_j} \right\} \right\} \\ & + \frac{\partial}{\partial x_i} \left\{ b_{T,ij}^{\alpha\beta}(x/\varepsilon) K_{\varepsilon, \delta} \left(\frac{\partial u_0^\beta}{\partial x_j} \right) \right\} \\ & + \varepsilon \frac{\partial}{\partial x_i} \left\{ a_{ij}^{\alpha\beta}(x/\varepsilon) \chi_{T,k}^{\beta\gamma}(x/\varepsilon) \frac{\partial}{\partial x_j} K_{\varepsilon, \delta} \left(\frac{\partial u_0^\gamma}{\partial x_k} \right) \right\}, \end{aligned} \quad (10.8)$$

where the function $b_{T,ij}^{\alpha\beta}$ is given in (9.1).

Proof. This follows by some direct algebraic manipulation, using $\mathcal{L}_\varepsilon(u_\varepsilon) = F = \mathcal{L}_0(u_0)$. □

Lemma 10.3. *Let $u_0 \in H^2(\Omega; \mathbb{R}^m)$, then*

$$\begin{aligned} & \frac{\partial}{\partial x_i} \left\{ b_{T,ij}^{\alpha\beta}(x/\varepsilon) K_{\varepsilon,\delta} \left(\frac{\partial u_0^\beta}{\partial x_j} \right) \right\} \\ &= \langle b_{T,ij}^{\alpha\beta} \rangle \frac{\partial}{\partial x_i} K_{\varepsilon,\delta} \left(\frac{\partial u_0^\beta}{\partial x_j} \right) + \frac{\partial}{\partial x_i} \left\{ T^{-2} \phi_{T,ij}^{\alpha\beta}(x/\varepsilon) K_{\varepsilon,\delta} \left(\frac{\partial u_0^\beta}{\partial x_j} \right) \right\} \\ & \quad - \frac{\partial}{\partial x_i} \left\{ \frac{\partial}{\partial x_i} h_{T,j}^{\alpha\beta}(x/\varepsilon) K_{\varepsilon,\delta} \left(\frac{\partial u_0^\beta}{\partial x_j} \right) \right\} \\ & \quad + \varepsilon \frac{\partial}{\partial x_i} \left\{ \left[\frac{\partial}{\partial x_k} (\phi_{T,ij}^{\alpha\beta})(x/\varepsilon) - \frac{\partial}{\partial x_i} (\phi_{T,kj}^{\alpha\beta})(x/\varepsilon) \right] \frac{\partial}{\partial x_k} K_{\varepsilon,\delta} \left(\frac{\partial u_0^\beta}{\partial x_j} \right) \right\}, \end{aligned}$$

where the function $\phi_{T,ij}^{\alpha\beta}(y)$ is defined by (9.2) and

$$h_{T,j}^{\alpha\beta}(y) = \frac{\partial}{\partial y_k} \phi_{T,kj}^{\alpha\beta}. \quad (10.9)$$

Proof. This follows from the identity

$$b_{T,ij}^{\alpha\beta} = \langle b_{T,ij}^{\alpha\beta} \rangle - \frac{\partial}{\partial y_k} \left(\frac{\partial}{\partial y_k} \phi_{T,ij}^{\alpha\beta} - \frac{\partial}{\partial y_i} \phi_{T,kj}^{\alpha\beta} \right) - \frac{\partial}{\partial y_i} \left(\frac{\partial}{\partial y_k} \phi_{T,kj}^{\alpha\beta} \right) + T^{-2} \phi_{T,ij}^{\alpha\beta}, \quad (10.10)$$

as well as the fact that the second term in the r.h.s. of (10.10) is skew-symmetric with respect to (i, k) . \square

The formulas in the previous two lemmas allow us to establish the following.

Lemma 10.4. *Let w_ε be the same as in Lemma 10.2, $T = \varepsilon^{-1} > 1$ and $2\varepsilon \leq \delta < 2$. Then, for any $\varphi \in H_0^1(\Omega; \mathbb{R}^m)$,*

$$\begin{aligned} & \left| \int_{\Omega} A(x/\varepsilon) \nabla w_\varepsilon \cdot \nabla \varphi \right| \\ & \leq C \left\{ \delta + \|\nabla \chi_T - \psi\|_{B^2} + T^{-1} \|\chi_T\|_{S_1^2} + T^{-2} \|\phi_T\|_{S_1^2} + T^{-1} \|\nabla \phi_T\|_{S_1^2} \right\} \\ & \quad \cdot \left\{ \|\nabla \varphi\|_{L^2(\Omega)} + \delta^{-1/2} \|\nabla \varphi\|_{L^2(\Omega_{4\delta})} \right\} \|u_0\|_{H^2(\Omega)}, \end{aligned} \quad (10.11)$$

where ψ is defined by (2.4) and the constant C depends only on A and Ω .

Proof. It follows from (10.8) that

$$\begin{aligned} \int_{\Omega} A(x/\varepsilon) \nabla w_\varepsilon \cdot \nabla \varphi &= - \int_{\Omega} \left\{ \widehat{a}_{ij}^{\alpha\beta} - a_{ij}^{\alpha\beta}(x/\varepsilon) \right\} \left\{ K_{\varepsilon,\delta} \left(\frac{\partial u_0^\beta}{\partial x_j} \right) - \frac{\partial u_0^\beta}{\partial x_j} \right\} \frac{\partial \varphi^\alpha}{\partial x_i} \\ & \quad - \varepsilon \int_{\Omega} a_{ij}^{\alpha\beta}(x/\varepsilon) \chi_{T,k}^{\beta\gamma}(x/\varepsilon) \frac{\partial}{\partial x_j} K_{\varepsilon,\delta} \left(\frac{\partial u_0^\gamma}{\partial x_k} \right) \frac{\partial \varphi^\alpha}{\partial x_i} \\ & \quad - \int_{\Omega} b_{T,ij}^{\alpha\beta}(x/\varepsilon) K_{\varepsilon,\delta} \left(\frac{\partial u_0^\beta}{\partial x_j} \right) \frac{\partial \varphi^\alpha}{\partial x_i}. \end{aligned} \quad (10.12)$$

Observe that

$$K_{\varepsilon,\delta}(\nabla u_0) - \nabla u_0 = S_\varepsilon(\eta_\delta \nabla u_0) - \eta_\delta \nabla u_0 + (\eta_\delta - 1) \nabla u_0, \quad (10.13)$$

and $\eta_\delta - 1 = 0$ in $\Omega \setminus \Omega_{2\delta}$. Thus, in view of (10.4) and Lemma 10.1, the first term in the r.h.s. of (10.12) is bounded by

$$\begin{aligned} & C \|S_\varepsilon(\eta_\delta \nabla u_0) - \eta_\delta \nabla u_0\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)} + C \|\nabla u_0\|_{L^2(\Omega_{2\delta})} \|\nabla \varphi\|_{L^2(\Omega_{2\delta})} \\ & \leq C \left\{ \varepsilon \|\nabla \varphi\|_{L^2(\Omega)} + \delta^{1/2} \|\nabla \varphi\|_{L^2(\Omega_{2\delta})} \right\} \|u_0\|_{H^2(\Omega)} \\ & \leq C \delta \left\{ \|\nabla \varphi\|_{L^2(\Omega)} + \delta^{-1/2} \|\nabla \varphi\|_{L^2(\Omega_{2\delta})} \right\} \|u_0\|_{H^2(\Omega)}. \end{aligned} \quad (10.14)$$

Next, note that

$$\frac{\partial}{\partial x_j} K_{\varepsilon, \delta} \left(\frac{\partial u_0^\gamma}{\partial x_k} \right) = S_\varepsilon \left(\eta_\delta \frac{\partial^2 u_0^\gamma}{\partial x_j \partial x_k} \right) + S_\varepsilon \left(\frac{\partial \eta_\delta}{\partial x_j} \frac{\partial u_0^\gamma}{\partial x_k} \right), \quad (10.15)$$

and the last term of (10.15) is zero in $\Omega \setminus \Omega_{4\delta}$, since $\delta > 2\varepsilon$. It follows from (10.3) and Lemma 10.1 that the second term in the r.h.s. of (10.12) is bounded by

$$\begin{aligned} & C \|\varepsilon \chi_T S_\varepsilon(\eta_\delta \nabla^2 u_0)\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)} + C \|\varepsilon \chi_T S_\varepsilon(\nabla \eta_\delta \nabla u_0)\|_{L^2(\Omega_{4\delta})} \|\nabla \varphi\|_{L^2(\Omega_{4\delta})} \\ & \leq C \|\varepsilon \chi_T\|_{S_1^2} \|\nabla^2 u_0\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)} \\ & \quad + C \delta^{-1} \|\varepsilon \chi_T\|_{S_1^2} \|\nabla u_0\|_{L^2(\Omega_{4\delta})} \|\nabla \varphi\|_{L^2(\Omega_{4\delta})} \\ & \leq C \left\{ \|\nabla \varphi\|_{L^2(\Omega)} + \delta^{-1/2} \|\nabla \varphi\|_{L^2(\Omega_{4\delta})} \right\} \|\varepsilon \chi_T\|_{S_1^2} \|u_0\|_{H^2(\Omega)}. \end{aligned} \quad (10.16)$$

It remains to estimate the third term in the r.h.s. of (10.12). To this end we use Lemma 10.3 to obtain

$$\begin{aligned} & \int_{\Omega} b_{T,ij}^{\alpha\beta}(x/\varepsilon) K_{\varepsilon, \delta} \left(\frac{\partial u_0^\beta}{\partial x_j} \right) \frac{\partial \varphi^\alpha}{\partial x_i} \\ & = \langle b_{T,ij}^{\alpha\beta} \rangle \int_{\Omega} K_{\varepsilon, \delta} \left(\frac{\partial u_0^\beta}{\partial x_j} \right) \frac{\partial \varphi^\alpha}{\partial x_i} + \int_{\Omega} T^{-2} \phi_{T,ij}^{\alpha\beta}(x/\varepsilon) K_{\varepsilon, \delta} \left(\frac{\partial u_0^\beta}{\partial x_j} \right) \frac{\partial \varphi^\alpha}{\partial x_i} \\ & \quad - \int_{\Omega} \frac{\partial}{\partial x_i} h_{T,j}^{\alpha\beta}(x/\varepsilon) K_{\varepsilon, \delta} \left(\frac{\partial u_0^\beta}{\partial x_j} \right) \frac{\partial \varphi^\alpha}{\partial x_i} \\ & \quad + \varepsilon \int_{\Omega} \left[\frac{\partial}{\partial x_k} (\phi_{T,ij}^{\alpha\beta})(x/\varepsilon) - \frac{\partial}{\partial x_i} (\phi_{T,kj}^{\alpha\beta})(x/\varepsilon) \right] \frac{\partial}{\partial x_k} K_{\varepsilon, \delta} \left(\frac{\partial u_0^\beta}{\partial x_j} \right) \frac{\partial \varphi^\alpha}{\partial x_i}. \end{aligned} \quad (10.17)$$

It follows from (2.5) and (10.2) that,

$$\left| \langle b_{T,ij}^{\alpha\beta} \rangle \int_{\Omega} K_{\varepsilon, \delta} \left(\frac{\partial u_0^\beta}{\partial x_j} \right) \frac{\partial \varphi^\alpha}{\partial x_i} \right| \leq C \|\nabla \chi_T - \psi\|_{B^2} \|\nabla \varphi\|_{L^2(\Omega)} \|u_0\|_{H^1(\Omega)}. \quad (10.18)$$

Also, the second and third terms in the r.h.s. of (10.17) are bounded by

$$C \left\{ T^{-2} \|\phi_T\|_{S_1^2} + \|\nabla h_T\|_{S_1^2} \right\} \|\nabla u_0\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)}, \quad (10.19)$$

while the last term is bounded by

$$C \|\varepsilon \nabla \phi_T\|_{S_1^2} \left\{ \|\nabla \varphi\|_{L^2(\Omega)} + \delta^{-1/2} \|\nabla \varphi\|_{L^2(\Omega_{4\delta})} \right\} \|u_0\|_{H^2(\Omega)}. \quad (10.20)$$

As a result, we obtain

$$\begin{aligned}
& \left| \int_{\Omega} b_{T,ij}^{\alpha\beta}(x/\varepsilon) K_{\varepsilon,\delta} \left(\frac{\partial u_0^\beta}{\partial x_j} \right) \frac{\partial \varphi^\alpha}{\partial x_i} \right| \\
& \leq C \left\{ \|\chi_T - \psi\|_{B^2} + T^{-2} \|\phi_T\|_{S_1^2} + T^{-1} \|\nabla \phi_T\|_{S_1^2} + \|\nabla h_T\|_{S_1^2} \right\} \\
& \quad \cdot \left\{ \|\nabla \varphi\|_{L^2(\Omega)} + \delta^{-1/2} \|\nabla \varphi\|_{L^2(\Omega_{4\delta})} \right\} \|u_0\|_{H^2(\Omega)}.
\end{aligned} \tag{10.21}$$

Finally, the estimate (10.11) follows by combining (10.12), (10.14), (10.16) and (10.21). The estimate (9.12) is also used here. \square

The next theorem provides an error estimate for u_ε in $H^1(\Omega)$.

Theorem 10.5. *Suppose that $A \in APW^2(\mathbb{R}^d)$ and satisfies (1.2). Let Ω be a bounded Lipschitz domain in \mathbb{R}^d and $0 < \varepsilon < 1$. Let $u_\varepsilon, u_0 \in H^1(\Omega; \mathbb{R}^m)$ be weak solutions of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$, $\mathcal{L}_0(u_0) = F$ in Ω , respectively. Assume that $u_\varepsilon = u_0$ on $\partial\Omega$ and $u_0 \in H^2(\Omega; \mathbb{R}^m)$. Then*

$$\|u_\varepsilon - u_0 - \varepsilon \chi_{T,k}^\beta(x/\varepsilon) K_{\varepsilon,\delta} \left(\frac{\partial u_0^\beta}{\partial x_k} \right)\|_{H_0^1(\Omega)} \leq C \delta^{1/2} \|u_0\|_{H^2(\Omega)}, \tag{10.22}$$

where $T = \varepsilon^{-1}$,

$$\delta = 2T^{-1} + \|\nabla \chi_T - \psi\|_{B^2} + T^{-1} \|\chi_T\|_{S_1^2} + T^{-2} \|\phi_T\|_{S_1^2} + T^{-1} \|\nabla \phi_T\|_{S_1^2}, \tag{10.23}$$

and C depends only on Ω and A .

Proof. This follows from Lemma 10.4 by letting $\varphi = w_\varepsilon$, where w_ε is defined by (10.7) with δ given by (10.23). Note that $w_\varepsilon \in H_0^1(\Omega; \mathbb{R}^m)$ and $\delta \geq 2\varepsilon$. \square

Theorem 10.6. *Let A , Ω , u_ε and u_0 be the same as in Theorem 10.5. We further assume that Ω is a bounded $C^{1,1}$ domain. Let δ^* be defined by (10.23), but with A replaced by A^* . Then*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C \{\delta + \delta^*\} \|u_0\|_{H^2(\Omega)}, \tag{10.24}$$

where δ is given by (10.23) and C depends only on A and Ω .

Proof. The theorem is proved by a duality argument, following the approach in [19]. Consider the Dirichlet problem,

$$\mathcal{L}_\varepsilon^*(v_\varepsilon) = G \quad \text{in } \Omega \quad \text{and} \quad v_\varepsilon = 0 \quad \text{on } \partial\Omega, \tag{10.25}$$

where $\mathcal{L}_\varepsilon^*$ is the adjoint operator of \mathcal{L}_ε . The corresponding homogenized problem of (10.25) is given by

$$\mathcal{L}_0^*(v_0) = G \quad \text{in } \Omega \quad \text{and} \quad v_0 = 0 \quad \text{on } \partial\Omega. \tag{10.26}$$

where \mathcal{L}_0^* is the adjoint operator of \mathcal{L}_0 . It is known that if Ω is a bounded $C^{1,1}$ domain and $G \in L^2(\Omega; \mathbb{R}^m)$, then the unique weak solution v_0 of (10.26) with constant coefficients is in $H^2(\Omega; \mathbb{R}^m)$ and satisfies the estimate

$$\|v_0\|_{H^2(\Omega)} \leq C \|G\|_{L^2(\Omega)}, \tag{10.27}$$

where C depends only on d, m, μ and Ω . Let

$$\rho_\varepsilon = v_\varepsilon - v_0 - \varepsilon \chi_{T,k}^{*\beta}(x/\varepsilon) K_{\varepsilon, \delta^* + 5\delta} \left(\frac{\partial v_0^\beta}{\partial x_k} \right), \quad (10.28)$$

where χ_T^* denotes the approximate corrector for operators $\{\mathcal{L}_\varepsilon^*\}$. It follows from Theorem 10.5 that

$$\|\rho_\varepsilon\|_{H_0^1(\Omega)} \leq C(\delta + \delta^*)^{1/2} \|v_0\|_{H^2(\Omega)}. \quad (10.29)$$

Next, we observe that to show estimate (10.24), it suffices to prove

$$\|w_\varepsilon\|_{L^2(\Omega)} \leq C\{\delta + \delta^*\} \|u_0\|_{H^2(\Omega)}, \quad (10.30)$$

where w_ε is defined by (10.7). This is because

$$\left\| \varepsilon \chi_{T,k}^{*\beta}(x/\varepsilon) K_{\varepsilon, \delta} \left(\frac{\partial u_0^\beta}{\partial x_k} \right) \right\|_{L^2(\Omega)} \leq C \delta \|u_0\|_{H^1(\Omega)}, \quad (10.31)$$

by (10.3). To prove (10.30), we use

$$\begin{aligned} \int_{\Omega} w_\varepsilon \cdot G &= \int_{\Omega} A(x/\varepsilon) \nabla w_\varepsilon \cdot \nabla v_\varepsilon \\ &= \int_{\Omega} A(x/\varepsilon) \nabla w_\varepsilon \cdot \nabla \rho_\varepsilon + \int_{\Omega} A(x/\varepsilon) \nabla w_\varepsilon \cdot \nabla v_0 \\ &\quad + \int_{\Omega} A(x/\varepsilon) \nabla w_\varepsilon \cdot \nabla \left(\varepsilon \chi_{T,k}^{*\beta}(x/\varepsilon) K_{\varepsilon, \delta^* + 5\delta} \left(\frac{\partial v_0^\beta}{\partial x_k} \right) \right). \end{aligned} \quad (10.32)$$

It follows from (10.22) and (10.29) that

$$\begin{aligned} \left| \int_{\Omega} A(x/\varepsilon) \nabla w_\varepsilon \cdot \nabla \rho_\varepsilon \right| &\leq C \|\nabla w_\varepsilon\|_{L^2(\Omega)} \|\nabla \rho_\varepsilon\|_{L^2(\Omega)} \\ &\leq C\{\delta + \delta^*\} \|u_0\|_{H^2(\Omega)} \|v_0\|_{H^2(\Omega)}. \end{aligned} \quad (10.33)$$

To handle the second integral in the r.h.s. of (10.32), we observe that $v_0 \in H_0^1(\Omega; \mathbb{R}^m)$ and thus by Lemma 10.4,

$$\begin{aligned} \left| \int_{\Omega} A(x/\varepsilon) \nabla w_\varepsilon \cdot \nabla v_0 \right| &\leq C\{\delta \|\nabla v_0\|_{L^2(\Omega)} + \delta^{1/2} \|\nabla v_0\|_{L^2(\Omega_{4\delta})}\} \|u_0\|_{H^2(\Omega)} \\ &\leq C \delta \|v_0\|_{H^2(\Omega)} \|u_0\|_{H^2(\Omega)}, \end{aligned} \quad (10.34)$$

where we have used Lemma 10.1 for the last inequality.

Finally, to bound the last integral in the r.h.s. of (10.32), we apply Lemma 10.4 again to obtain

$$\begin{aligned} &\left| \int_{\Omega} A(x/\varepsilon) \nabla w_\varepsilon \cdot \nabla \left(\varepsilon \chi_{T,k}^{*\beta}(x/\varepsilon) K_{\varepsilon, \delta^* + 5\delta} \left(\frac{\partial v_0^\beta}{\partial x_k} \right) \right) \right| \\ &\leq C \|u_0\|_{H^2(\Omega)} \cdot \left\{ \delta \left\| \nabla \left(\varepsilon \chi_{T,k}^{*\beta}(x/\varepsilon) K_{\varepsilon, \delta^* + 5\delta} \left(\frac{\partial v_0^\beta}{\partial x_k} \right) \right) \right\|_{L^2(\Omega)} \right. \\ &\quad \left. + \delta^{1/2} \left\| \nabla \left(\varepsilon \chi_{T,k}^{*\beta}(x/\varepsilon) K_{\varepsilon, \delta^* + 5\delta} \left(\frac{\partial v_0^\beta}{\partial x_k} \right) \right) \right\|_{L^2(\Omega_{4\delta})} \right\}. \end{aligned} \quad (10.35)$$

By (10.29) and the energy estimate,

$$\begin{aligned} \left\| \nabla \left(\varepsilon \chi_{T,k}^{*\beta}(x/\varepsilon) K_{\varepsilon, \delta^* + 5\delta} \left(\frac{\partial v_0^\beta}{\partial x_k} \right) \right) \right\|_{L^2(\Omega)} &\leq \|\rho_\varepsilon\|_{H^1(\Omega)} + \|v_\varepsilon\|_{H^1(\Omega)} + \|v_0\|_{H^1(\Omega)} \\ &\leq C \|v_0\|_{H^2(\Omega)}. \end{aligned}$$

Also note that

$$K_{\varepsilon, \delta^* + 5\delta} \left(\frac{\partial v_0^\beta}{\partial x_k} \right) = 0 \quad \text{in } \Omega_{4\delta}.$$

As a result, it follows from (10.35) that

$$\left| \int_{\Omega} A(x/\varepsilon) \nabla w_\varepsilon \cdot \nabla \left(\varepsilon \chi_{T,k}^{*\beta}(x/\varepsilon) K_{\varepsilon, \delta^* + 5\delta} \left(\frac{\partial v_0^\beta}{\partial x_k} \right) \right) \right| \leq C \delta \|v_0\|_{H^2(\Omega)} \|u_0\|_{H^2(\Omega)}. \quad (10.36)$$

This, together with (10.32), (10.33) and (10.34), leads to

$$\begin{aligned} \left| \int_{\Omega} w_\varepsilon \cdot G \right| &\leq C \{\delta + \delta^*\} \|v_0\|_{H^2(\Omega)} \|u_0\|_{H^2(\Omega)} \\ &\leq C \{\delta + \delta^*\} \|G\|_{L^2(\Omega)} \|u_0\|_{H^2(\Omega)}. \end{aligned} \quad (10.37)$$

Therefore, by duality,

$$\|w_\varepsilon\|_{L^2(\Omega)} \leq C \{\delta + \delta^*\} \|u_0\|_{H^2(\Omega)}, \quad (10.38)$$

which completes the proof of Theorem 10.6. \square

Proof of Theorem 1.4. Let δ, δ^* be the same as in Theorem 10.6. Let $\Theta_{k,\sigma}(T)$ denote the integral in the r.h.s. of (1.8). It follows from Theorems 1.1, 9.5 and 9.6 that

$$\begin{aligned} \delta &\leq C_\sigma \{ \|\nabla \chi_T - \psi\|_{B^2} + T^{-1} \Theta_{k,\sigma}(T) \}, \\ \delta^* &\leq C_\sigma \{ \|\nabla \chi_T^* - \psi^*\|_{B^2} + T^{-1} \Theta_{k,\sigma}(T) \}, \end{aligned}$$

for any $k \geq 1$ and $\sigma \in (0, 1)$, where C_σ depends only on σ, k and A . This, together with Theorem 10.6, gives the estimate (1.13) in Theorem 1.4.

Now suppose that the condition (1.10) holds for some $\alpha > 1$ and $k \geq 1$. Then, by Theorem 1.2,

$$\|\chi_T\|_{S_1^2} + \|\chi_T^*\|_{S_1^2} \leq C.$$

To see (1.14), we note that by the proof of Theorem 6.6 in [17, p.1590],

$$\|\nabla \chi_T - \psi\|_{B^2} + \|\nabla \chi_T^* - \psi^*\|_{B^2} \leq C \sum_{j=1}^{\infty} \left\{ \left\| \frac{\chi_{2^j T}}{2^j T} \right\|_{B^2} + \left\| \frac{\chi_{2^j T}^*}{2^j T} \right\|_{B^2} \right\} \leq \frac{C}{T},$$

for any $T > 1$. Finally, using the same argument as in the case of χ_T , we may show that

$$T^{-1} \|\phi_T\|_{S_1^2} + \|\nabla \phi_T\|_{S_1^2} + T^{-1} \|\phi_T^*\|_{S_1^2} + \|\nabla \phi_T^*\|_{S_1^2} \leq C.$$

As a result, we obtain $\delta + \delta^* \leq C T^{-1}$. In view of Theorem 10.6, this gives the $O(\varepsilon)$ estimate (1.14). \square

11 Lipschitz estimates at large scale

In this section we establish an interior L^2 -based Lipschitz estimate at large scale under a general condition: there exists a nonnegative increasing function $\eta(t)$ on $[0, 1]$ with the Dini property

$$\int_0^1 \frac{\eta(t)}{t} dt < \infty, \quad (11.1)$$

such that

$$\|u_\varepsilon - u_0\|_{L^2(B(x_0, 1))} \leq [\eta(\varepsilon)]^2 \|u_0\|_{H^2(B(x_0, 1))}, \quad (11.2)$$

whenever $u_\varepsilon, u_0 \in H^1(B(x_0, 1); \mathbb{R}^m)$, $\mathcal{L}_\varepsilon(u_\varepsilon) = \mathcal{L}_0(u_0)$ in $B(x_0, 1)$ and $u_\varepsilon = u_0$ on $\partial B(x_0, 1)$, for some $x_0 \in \mathbb{R}^d$ and $0 < \varepsilon < 1$. We note that by rescaling, (11.2) continues to hold if $B(0, 1)$ is replaced by $B(0, r)$ for $1 \leq r \leq 2$.

Theorem 11.1. *Suppose that $A \in B^2(\mathbb{R}^d)$ and satisfies the ellipticity condition (1.2). Also assume that conditions (11.1)-(11.2) hold. Let $u_\varepsilon \in H^1(B(x_0, R); \mathbb{R}^m)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $B(x_0, R)$ for some $x_0 \in \mathbb{R}^d$ and $R > \varepsilon$. Then for any $\varepsilon \leq r \leq R$ and $\sigma \in (0, 1)$,*

$$\begin{aligned} & \left(\int_{B(x_0, r)} |\nabla u_\varepsilon|^2 \right)^{1/2} \\ & \leq C_\sigma \left\{ \left(\int_{B(x_0, R)} |\nabla u_\varepsilon|^2 \right)^{1/2} + \sup_{\substack{x \in B(x_0, R/2) \\ \varepsilon \leq t \leq R/2}} t \left(\frac{R}{t} \right)^\sigma \left(\int_{B(x, t)} |F|^2 \right)^{1/2} \right\}, \end{aligned} \quad (11.3)$$

where C_σ depends only on d, m, μ, σ , and the function η in (11.2).

The proof of Theorem 11.1 is based on a general approach originated in [4] and further developed in [3, 2, 16] for Lipschitz estimates. The argument in this section follows closely that in [16], where the large-scale boundary Lipschitz estimates for systems of linear elasticity with bounded measurable periodic coefficients $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ are obtained for $F \in L_{\text{loc}}^p$, $p > d$. However, in order to apply the estimates to χ_T for elliptic systems with bounded measurable a.p. coefficients, we need to consider the case where $F \in L_{\text{loc}}^2$. As a result, modifications of the argument in [3, 16] are needed for the proof of Theorem 1.3. Notice that if $F \in L_{\text{loc}}^p$ for some $p > d$, the second term in the r.h.s. of (11.3) with $\sigma = \frac{d}{p}$ is bounded by

$$C R \left(\int_{B(x_0, R)} |F|^p \right)^{1/p}.$$

We begin with a lemma that utilizes the condition (11.2).

Lemma 11.2. *Assume A satisfies the same conditions as in Theorem 11.1. Let $u_\varepsilon \in H^1(B(0, 2); \mathbb{R}^m)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $B(0, 2)$, where $0 < \varepsilon < 1$ and $F \in L^2(B(0, 2); \mathbb{R}^m)$. Then there exists $v \in H^1(B(0, 1); \mathbb{R}^m)$ such that $\mathcal{L}_0(v) = F$ in $B(0, 1)$ and*

$$\|u_\varepsilon - v\|_{L^2(B(0, 1))} \leq C \eta(\varepsilon) \left\{ \|u_\varepsilon\|_{L^2(B(0, 2))} + \|F\|_{L^2(B(0, 2))} \right\}, \quad (11.4)$$

where C depends only on d, m and μ .

Proof. By Caccioppoli's inequality,

$$\int_{B(0,3/2)} |\nabla u_\varepsilon|^2 \leq C \int_{B(0,2)} |u_\varepsilon|^2 + C \int_{B(0,2)} |F|^2.$$

It follows by the co-area formula that there exists some $r_0 \in (1, 3/2)$ such that $u_\varepsilon \in H^1(\partial B(0, r_0); \mathbb{R}^m)$ and

$$\int_{\partial B(0, r_0)} |\nabla u_\varepsilon|^2 \leq C \int_{B(0,2)} |u_\varepsilon|^2 + C \int_{B(0,2)} |F|^2. \quad (11.5)$$

Let $f = u_\varepsilon|_{\partial B(0, r_0)}$. We choose $g_\delta \in H^{3/2}(\partial B(0, r_0); \mathbb{R}^m)$ such that

$$\begin{aligned} \|g_\delta - f\|_{H^{1/2}(\partial B(0, r_0))} &\leq C \delta^{1/2} \|f\|_{H^1(\partial B(0, r_0))}, \\ \|g_\delta\|_{H^{3/2}(\partial B(0, r_0))} &\leq C \delta^{-1/2} \|f\|_{H^1(\partial B(0, r_0))}. \end{aligned} \quad (11.6)$$

Let v_ε be the weak solution of $\mathcal{L}_\varepsilon(v_\varepsilon) = F$ in $B(0, r_0)$ with $v_\varepsilon = g_\delta$ on $\partial B(0, r_0)$, and v the weak solution of $\mathcal{L}_0(v) = F$ in $B(0, r_0)$ with $v = g_\delta$ on $\partial B(0, r_0)$. Then

$$\begin{aligned} \|u_\varepsilon - v\|_{L^2(B(0,1))} &\leq \|u_\varepsilon - v_\varepsilon\|_{L^2(B(0,1))} + \|v_\varepsilon - v\|_{L^2(B(0,1))} \\ &\leq \|u_\varepsilon - v_\varepsilon\|_{H^1(B(0, r_0))} + \|v_\varepsilon - v\|_{L^2(B(0, r_0))} \\ &\leq C \|f - g_\delta\|_{H^{1/2}(\partial B(0, r_0))} + C [\eta(\varepsilon)]^2 \|v\|_{H^2(B(0, r_0))} \\ &\leq C \delta^{1/2} \|f\|_{H^1(\partial B(0, r_0))} + C [\eta(\varepsilon)]^2 \delta^{-1/2} \|f\|_{H^1(\partial B(0, r_0))} \\ &\leq C \left\{ \delta^{1/2} + [\eta(\varepsilon)]^2 \delta^{-1/2} \right\} \left\{ \|u_\varepsilon\|_{L^2(B(0,2))} + \|F\|_{L^2(B(0,2))} \right\}, \end{aligned}$$

where we have used the condition (11.2) for the third inequality, (11.6) for the fourth and (11.5) for the last. Estimate (11.4) now follows by letting $\delta = [\eta(\varepsilon)]^2$. \square

Lemma 11.3. *Assume A satisfies the same conditions as in Theorem 11.1. Let $\varepsilon \leq r < 1$. Let $u_\varepsilon \in H^1(B(0, 2r); \mathbb{R}^m)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $B(0, 2r)$ for some $F \in L^2(B(0, 2r); \mathbb{R}^m)$. Then there exists $v \in H^1(B(0, r); \mathbb{R}^m)$ such that $\mathcal{L}_0(v) = F$ in $B(0, r)$ and*

$$\begin{aligned} &\left(\int_{B(0,r)} |u_\varepsilon - v|^2 \right)^{1/2} \\ &\leq C \eta(\varepsilon/r) \left\{ \left(\int_{B(0,2r)} |u_\varepsilon|^2 \right)^{1/2} + r^2 \left(\int_{B(0,2r)} |F|^2 \right)^{1/2} \right\}, \end{aligned} \quad (11.7)$$

where C depends only on d , m and μ .

Proof. Note that if $v(x) = u_\varepsilon(rx)$, then $\mathcal{L}_{\frac{\varepsilon}{r}}(v)(x) = r^2 F(rx)$. As a result, the estimate (11.7) follows readily from Lemma 11.2 by rescaling. \square

The next lemma gives a regularity property for solutions of elliptic systems with constant coefficients.

Lemma 11.4. *Let v be a weak solution of $\mathcal{L}_0(v) = F$ in $B(0, r)$ for some $F \in L^2(B(0, r); \mathbb{R}^m)$. Then, for any $0 < t < r/4$,*

$$\begin{aligned} & \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \frac{1}{t} \left(\int_{B(0, t)} |v - Mx - q|^2 \right)^{1/2} \\ & \leq C \left(\frac{t}{r} \right) \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \frac{1}{r} \left(\int_{B(0, r)} |v - Mx - q|^2 \right)^{1/2} \\ & \quad + C \log \left(\frac{r}{t} \right) \sup_{x \in B(0, \frac{r}{2})} t \left(\int_{B(x, t)} |F|^2 \right)^{1/2}, \end{aligned} \quad (11.8)$$

where C depends only on d , m and μ .

Proof. By rescaling we may assume $r = 1$. We may also assume that $0 < t < 1/100$, as the case $1/100 \leq t \leq 1$ is trivial. Let $\Gamma_0(x)$ denote the matrix of fundamental solutions for the operator \mathcal{L}_0 with constant coefficients. Let $\varphi \in C_0^1(B(0, 1/2))$ such that $\varphi = 1$ in $B(0, 3/8)$. Using the representation by fundamental solutions, we may write $v(x) = w(x) + I(x)$ for $x \in B(0, 1/4)$, where

$$w(x) = \int_{B(0, 1/2)} \Gamma_0(x - y) F(y) \varphi(y) dy$$

and the function $I(x)$ satisfies

$$|\nabla^2 I(x)| \leq C \left\{ \|v\|_{L^2(B(0, 1))} + \|F\|_{L^2(B(0, 1/2))} \right\}.$$

Note that for $x \in B(0, t)$, where $0 < t < 1/100$,

$$|\nabla^2 w(x)| \leq \left| \nabla_x^2 \int_{B(0, 2t)} \Gamma_0(x - y) F(y) dy \right| + C \int_{2t \leq |y| \leq (1/2)} \frac{|F(y)|}{|y|^d} dy, \quad (11.9)$$

where we have used the estimate $|\nabla^2 \Gamma_0(x)| \leq C|x|^{-d}$.

Next, we observe that the second term in the r.h.s. of (11.9) is bounded by

$$C \log \left(\frac{1}{t} \right) \sup_{x \in B(0, \frac{1}{2})} \left(\int_{B(x, t)} |F|^2 \right)^{1/2}.$$

To handle the first term in the r.h.s. of (11.9), we use the singular integral estimates. As a result we obtain

$$\left(\int_{B(0, t)} |\nabla^2 w| \right)^{1/2} \leq C \log \left(\frac{1}{t} \right) \sup_{x \in B(0, \frac{1}{2})} \left(\int_{B(x, t)} |F|^2 \right)^{1/2}.$$

Finally, we note that

$$\begin{aligned}
& \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \frac{1}{t} \left(\int_{B(0,t)} |v - Mx - q|^2 \right)^{1/2} \leq C t \left(\int_{B(0,t)} |\nabla^2 v|^2 \right)^{1/2} \\
& \leq C t \left(\int_{B(0,t)} |\nabla^2 w|^2 \right)^{1/2} + C t \left(\int_{B(0,t)} |\nabla^2 I|^2 \right)^{1/2} \\
& \leq C t \left(\int_{B(0,1)} |v|^2 \right)^{1/2} + C \log \left(\frac{1}{t} \right) \sup_{x \in B(0, \frac{1}{2})} t \left(\int_{B(x,t)} |F|^2 \right)^{1/2}.
\end{aligned}$$

Since $\mathcal{L}_0(Mx + q) = 0$, we may replace v in the inequalities above by $v - Mx - q$ for any $M \in \mathbb{R}^{m \times d}$ and $q \in \mathbb{R}^m$. This gives the estimate (11.8). \square

Lemma 11.5. Fix $\sigma \in (0, 1)$. Let $u_\varepsilon \in H^1(B(0, 1); \mathbb{R}^m)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $B(0, 1)$, where $0 < \varepsilon < 1$ and $F \in L^2(B(0, 1); \mathbb{R}^m)$. Define

$$H(r) = \frac{1}{r} \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \left(\int_{B(0,r)} |u_\varepsilon - Mx - q|^2 \right)^{1/2} + \sup_{\substack{x \in B(0, 1/2) \\ \varepsilon \leq t \leq r/2}} t \left(\frac{r}{t} \right)^\sigma \left(\int_{B(x,t)} |F|^2 \right)^{1/2}, \quad (11.10)$$

and

$$\Psi(r) = \inf_{q \in \mathbb{R}^m} \frac{1}{r} \left(\int_{B(0,2r)} |u_\varepsilon - q|^2 \right)^{1/2} + r \left(\int_{B(0,2r)} |F|^2 \right)^{1/2}. \quad (11.11)$$

Then there exists $\theta \in (0, 1/4)$, depending only on d, m, σ and μ , such that

$$H(\theta r) \leq (1/2)H(r) + C\eta(\varepsilon/r)\Psi(r) \quad (11.12)$$

for any $r \in [\theta^{-1}\varepsilon, 1/2]$.

Proof. Let $r \in [\theta^{-1}\varepsilon, 1/2]$, where $\theta \in (0, 1/4)$ is to be determined. Let v be the solution of

$\mathcal{L}_0(v) = F$ in $B(0, r)$, given by Lemma 11.3. Observe that by Lemmas 11.4 and 11.3,

$$\begin{aligned}
H(\theta r) &\leq \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \frac{1}{\theta r} \left(\int_{B(0, \theta r)} |v - Mx - q|^2 \right)^{1/2} + \sup_{\substack{x \in B(0, 1/2) \\ \varepsilon \leq t \leq \theta r}} t \left(\frac{\theta r}{t} \right)^\sigma \left(\int_{B(x, t)} |F|^2 \right)^{1/2} \\
&\quad + \frac{1}{\theta r} \left(\int_{B(0, \theta r)} |u_\varepsilon - v|^2 \right)^{1/2} \\
&\leq C\theta \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \left(\int_{B(0, r)} |v - Mx - q|^2 \right)^{1/2} + \theta^\sigma \sup_{\substack{x \in B(0, 1/2) \\ \varepsilon \leq t \leq r/2}} t \left(\frac{r}{t} \right)^\sigma \left(\int_{B(x, t)} |F|^2 \right)^{1/2} \\
&\quad + C\theta \log(\theta^{-1}) \sup_{x \in B(0, r/2)} r \left(\int_{B(x, \theta r)} |F|^2 \right)^{1/2} + \frac{C_\theta}{r} \left(\int_{B(0, r)} |u_\varepsilon - v|^2 \right)^{1/2} \\
&\leq C \{ \theta + \theta^\sigma + \theta^\sigma \log(\theta^{-1}) \} H(r) + \frac{C_\theta}{r} \left(\int_{B(0, r)} |u_\varepsilon - v|^2 \right)^{1/2} \\
&\leq C \{ \theta + \theta^\sigma + \theta^\sigma \log(\theta^{-1}) \} H(r) \\
&\quad + \frac{C_\theta}{r} \eta(\varepsilon/r) \left\{ \left(\int_{B(0, 2r)} |u_\varepsilon|^2 \right)^{1/2} + r^2 \left(\int_{B(0, 2r)} |F|^2 \right)^{1/2} \right\}.
\end{aligned}$$

We now choose $\theta \in (0, 1/4)$ so small that $C \{ \theta + \theta^\sigma + \theta^\sigma \log(\theta^{-1}) \} \leq (1/2)$. Since the inequalities above also hold for $u_\varepsilon - q$ with any $q \in \mathbb{R}^m$, we obtain the estimate (11.12). \square

The next lemma was proved in [16].

Lemma 11.6. *Let $H(r)$ and $h(r)$ be two nonnegative continuous functions on the interval $(0, 1]$. Let $0 < \varepsilon < (1/4)$. Suppose that there exists a constant C_0 such that*

$$\begin{cases} \max_{r \leq t \leq 2r} H(t) \leq C_0 H(2r), \\ \max_{r \leq t, s \leq 2r} |h(t) - h(s)| \leq C_0 H(2r), \end{cases} \quad (11.13)$$

for any $r \in [\varepsilon, 1/2]$. We further assume that

$$H(\theta r) \leq (1/2)H(r) + C_0 \omega(\varepsilon/r) \{ H(2r) + h(2r) \}, \quad (11.14)$$

for any $r \in [\theta^{-1}\varepsilon, 1/2]$, where $\theta \in (0, 1/4)$ and ω is a nonnegative increasing function $[0, 1]$ such that $\omega(0) = 0$ and

$$\int_0^1 \frac{\omega(t)}{t} dt < \infty. \quad (11.15)$$

Then

$$\max_{\varepsilon \leq r \leq 1} \{ H(r) + h(r) \} \leq C \{ H(1) + h(1) \}, \quad (11.16)$$

where C depends only on C_0 , θ , and ω .

We are now in a position to give the proof of Theorem 11.1.

Proof of Theorem 11.1. By translation and dilation we may assume that $x_0 = 0$ and $R = 1$. Thus u_ε is a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $B(0, 1)$ and $0 < \varepsilon < (1/2)$. Let $H(r)$ be defined by (11.10). Let $h(r) = |M_r|$, where $M_r \in \mathbb{R}^{m \times d}$ is a matrix such that

$$\inf_{q \in \mathbb{R}^m} \frac{1}{r} \left(\int_{B(0,r)} |u_\varepsilon - M_r x - q|^2 \right)^{1/2} = \inf_{\substack{M \in \mathbb{R}^{m \times d} \\ q \in \mathbb{R}^m}} \frac{1}{r} \left(\int_{B(0,r)} |u_\varepsilon - Mx - q|^2 \right)^{1/2}.$$

As in [16], it follows that if $t, s \in [r, 2r]$,

$$|h(t) - h(s)| \leq |M_t - M_s| \leq C \left\{ H(t) + H(s) \right\} \leq CH(2r).$$

Also, if $\Psi(r)$ is defined by (11.11), then

$$\Psi(r) \leq H(2r) + h(2r).$$

By Lemma 11.5 we see that

$$H(\theta r) \leq (1/2)H(r) + C\eta(\varepsilon/r) \left\{ H(2r) + h(2r) \right\},$$

for any $r \in [\theta^{-1}\varepsilon, 1/2]$, where $\eta(t)$ satisfies the Dini condition (11.1). This allows us to apply Lemma 11.6 and obtain

$$\begin{aligned} \inf_{q \in \mathbb{R}^m} \frac{1}{r} \left(\int_{B(0,r)} |u_\varepsilon - q|^2 \right)^{1/2} &\leq C \left\{ H(r) + h(r) \right\} \leq C \left\{ H(1) + h(1) \right\} \\ &\leq C \left\{ \left(\int_{B(0,1)} |\nabla u_\varepsilon|^2 \right)^{1/2} + \sup_{\substack{x \in B(0,1/2) \\ \varepsilon \leq t \leq 1/2}} t^{1-\sigma} \left(\int_{B(x,t)} |F|^2 \right)^{1/2} \right\}. \end{aligned}$$

Hence, by Caccioppoli's inequality,

$$\left(\int_{B(0,r)} |\nabla u_\varepsilon|^2 \right)^{1/2} \leq C \left\{ \left(\int_{B(0,1)} |\nabla u_\varepsilon|^2 \right)^{1/2} + \sup_{\substack{x \in B(0,1/2) \\ \varepsilon \leq t \leq 1/2}} t^{1-\sigma} \left(\int_{B(x,t)} |F|^2 \right)^{1/2} \right\},$$

for any $r \in [\varepsilon, 1/2]$. The proof is complete. \square

Finally, we give the proof of Theorem 1.3

Proof of Theorem 1.3. Note that by (1.11),

$$\inf_{1 \leq L \leq t} \left\{ \rho_k(L, L) + \exp \left(-\frac{ct^2}{L^2} \right) \right\} \leq C \left\{ \left\{ \log(t+1) \right\}^{-\alpha} + \exp(-ct) \right\}.$$

Using

$$\int_2^T \frac{dt}{(\log t)^{\alpha} t^{\sigma}} \leq C_{\sigma, \alpha} T^{1-\sigma} (\log T)^{-\alpha}$$

for any $T \geq 2$, where $\sigma \in (0, 1)$ and $\alpha > 0$, we see that

$$\frac{1}{T} \int_1^T \inf_{1 \leq L \leq t} \left\{ \rho_k(L, L) + \exp \left(-\frac{ct^2}{L^2} \right) \right\} \left(\frac{T}{t} \right)^{\sigma} dt \leq C \left\{ \log(T+1) \right\}^{-\alpha}.$$

In view of Theorem 1.1, this yields

$$T^{-1}\|\chi_T\|_{S_1^2} \leq C\{\log(T+1)\}^{-\alpha},$$

and

$$\|\nabla\chi_T - \psi\|_{B^2} \leq C \sum_{j=1}^{\infty} (2^j T)^{-1} \|\chi_{2^j T}\|_{S_1^2} \leq C\{\log(T+1)\}^{1-\alpha}.$$

The same argument also gives the estimate for the adjoint operator,

$$\|\nabla\chi_T^* - \psi^*\|_{B^2} \leq C\{\log(T+1)\}^{1-\alpha}.$$

It then follows from Theorem 1.4 that the condition (11.2) holds for

$$\eta(t) = C\{\log(2/t)\}^{\frac{1-\alpha}{2}}.$$

Since $\alpha > 3$, the function $\eta(t)$ satisfies the Dini condition (11.1). As a result, Theorem 11.1 holds for the operator \mathcal{L}_ε .

Finally, let

$$u(x) = \chi_{T,j}^\beta(x) + P_j^\beta(x).$$

Then $\mathcal{L}_1(u) = -T^{-2}\chi_{T,j}^\beta$ in \mathbb{R}^d . It follows from Theorem 11.1 with $r = \varepsilon = 1$ and $R = T$ that

$$\left(\int_{B(x_0,1)} |\nabla u|^2 \right)^{1/2} \leq C \left(\int_{B(x_0,T)} |\nabla u|^2 \right)^{1/2} + CT^{-1}\|\chi_T\|_{S_1^2} \leq C,$$

for any $x_0 \in \mathbb{R}^d$. Since $|\nabla\chi_{T,j}^\beta| \leq |\nabla u| + C$, this gives the estimate (1.12). \square

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